Fuzzily Determined Interval Matrix Games

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Abstract

Matrix games have been widely used in decision making systems. In practice, for the same strategies players take, the corresponding payoffs may be within certain ranges rather than exact values. To model such uncertainty in matrix games, we consider interval-valued game matrices in this paper; and extend the results of classical strictly-determined matrix games to fuzzily determined interval matrix games.

1 Introduction

1.1 Matrix game

Game theory had its beginnings in the 1920s, and significantly advanced at Princeton University through the work of John Nash [1] and [6]. The simplest game is a zero sum one involving only two players. An $m \times n$ matrix $G = \{g_{ij}\}_{m\times n}$ may be used to model such a two-person zero-sum game. If the row player R uses his i^{th} strategy (row) and the column player C selects her j^{th} choice (column), then R wins (and subsequently C loses) the amount g_{ij}. The objective of R is to maximize his gain while C tries to minimize her loss.

Example 1: A game is described by the matrix

$$G = \begin{bmatrix} 0 & 6 & -2 & -4 \\ 5 & 2 & 1 & 3 \\ -8 & -1 & 0 & 20 \end{bmatrix} \tag{1}$$

In the game above, the players R and C have three and four possible strategies, respectively. If R chooses his first strategy and C chooses her second, then R wins $g_{12} = 6$ (C loses 6). If R chooses his third strategy and C chooses her first, then R wins $g_{31} = -8$ (R loses 8, C wins 8). In this paper we restrict our attention to such two-person zero-sum games.

1.2 Strictly Determined Matrix Game

If there exists a g_{ij} in a classical $m \times n$ game matrix G such that g_{ij} is simultaneously the minimum value of the i^{th} row and the maximum value of the j^{th} column of G then g_{ij} is called a *saddle value* of the game. If a matrix game has a saddle value it is said to be *strictly determined*. It is well known, [1] and [6], that the optimal strategies for both R and C in a strictly determined game are:

- R should choose any row containing a saddle value, and

- C should choose any column containing a saddle value.

A saddle value is also called the value of the (strictly determined) game. In (1), g_{23} is simultaneously the minimum of the second row and the maximum of the third column. Hence the game is strictly determined and its value is $g_{23} = 1$. The knowledge of an opponents move provides no advantage since the payoff will always be a saddle value in a strictly determined game.

1.3 Motivations of this work

Matrix games have many useful applications, especially in decision making systems. However, in real world applications, due to certain forms of uncertainly, outcomes of a matrix game may not be a fixed number even though the players do not change their strategies. Hence, fuzzy games have been studied [3, 5, 7]. By noticing the fact that the payoffs may only vary within a designated range for fixed strategies, we propose to use an interval-valued matrix, whose entries are closed intervals, to model such kind of uncertainty.

Throughout the rest of this paper we will use boldface letters to denote (closed and bounded) intervals. For example, $\mathbf{x}$ is an interval. Its greatest lower bound and the least upper bound are denoted by $\underline{x}$ and $\overline{x}$, respectively. We use uppercase letters to denote general matrices. An boldface upper case letter will represent an interval-valued matrix.

In this paper we assume that the intervals in the game matrix $\mathbf{G}$ are closed and bounded intervals of real numbers, and represent uniformly distributed possible payoffs. This means that each $x \in \mathbf{g}_{ij}$ has the same probability as the outcome/payoff.

Definition 2: Let $\mathbf{G} = \{\mathbf{g}_{ij}\}$ be an $m \times n$ interval-valued matrix. The matrix $\mathbf{G}$ defines a *zero-sum interval matrix game* provided whenever the row player R uses his i^{th} strategy and the column player C selects her j^{th} strategy, then R wins and C correspondingly loses a common $x \in \mathbf{g}_{ij}$.

Example 3: Consider the following interval game matrix:

$$\mathbf{G} = \begin{bmatrix} [0,1] & [6,7] & [-2,0] & [-4,-2] \\ [5,6] & [2,7] & [1,3] & [3,3] \\ [-8,-5] & [-1,0] & [0,\ 0] & [20,25] \end{bmatrix} \quad (2)$$

In this game, if R chooses row one and C selects column two, then R wins an amount $x \in [6, 7]$ (C loses the same x that R wins).

In this paper, we attempt to extend results of classical matrix games into interval-valued games. In order to accomplish this, we define fuzzy binary relational operators for intervals in section 2. We then study crisply determined and fuzzily determined interval games in section 3 and 4. We conclude the paper with section 5.

2 Comparing Intervals

In order to compare strategies and payoffs for an interval game matrix, we need to define a notion of interval inequality (both $\leq$ and $\geq$) that corresponds to an intuitive notion of a "better possible" outcome/payoff. Let $\mathbf{x}$ and $\mathbf{y}$ be two non-empty intervals. We will consider their relationship in the following different cases:

- Case 1: $\mathbf{x} \cap \mathbf{y} = \emptyset$ and $\overline{x} < \underline{y}$ (see Figure 1). In this case, every possible payoff value from $\mathbf{y}$ exceeds all of the possible payoffs from $\mathbf{x}$. Therefore, we say that $\mathbf{x} < \mathbf{y}$ and $\mathbf{y} > \mathbf{x}$ crisply, which corresponds to the traditional definition of comparison used in interval computations [4].

- Case 2: $\mathbf{x} = \mathbf{y}$. We then define the crisp inequalities $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \leq \mathbf{y}$, again paralleling common usage of existing interval inequality comparisons.

- Case 3: $\mathbf{x} \cap \mathbf{y} \neq \emptyset$ and $\mathbf{x} \neq \mathbf{y}$.

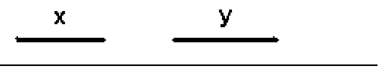

Figure 1: Non-overlapping intervals, $\mathbf{x} < \mathbf{y}$

- Case 3.1 $\mathbf{x} \not\subset \mathbf{y}$, and $\mathbf{y} \not\subset \mathbf{x}$. Without loss of generality, we may assume $\underline{x} < \underline{y} \leq \overline{x} < \overline{y}$, see Figure 2. Then, for any given $x \in [\underline{x}, \underline{y})$, we have $x < \mathbf{y}$(i.e. x is less than every value/payoff in $\mathbf{y}$). If $x \in [\underline{y}, \overline{x}]$, the value in both $\mathbf{x}$ and $\mathbf{y}$. Therefore, we define $\mathbf{x} \leq \mathbf{y}$ crisply for this case as $\mathbf{x}$ offers no larger payoff than what is possible in $\mathbf{y}$. Similarly, if $y \in (\overline{x}, \overline{y}]$ then y exceeds every value in $\mathbf{x}$. So, we also define the crisp inequality $\mathbf{y} \geq \mathbf{x}$. Both of these comparisons also mirror existing practice in interval computing.

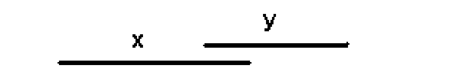

Figure 2: Overlapping intervals, $\mathbf{x} \leq \mathbf{y}$

- Case 3.2: $\mathbf{x} \subset \mathbf{y}$ (see Figure 3). (An analogous definition for $\mathbf{y} \subset \mathbf{x}$ is implied.) We can assume that $\mathbf{y}$ is a nontrivial interval. Here, we need use the width function $w(\ )$ defined as $w(\mathbf{y}) = \overline{y} - \underline{y}$. As $\mathbf{x}$ is a proper subset of $\mathbf{y}$ we know that $w(\mathbf{y}) - w(\mathbf{x}) > 0$. We define $\mathbf{x} \leq \mathbf{y}$ in terms of fuzzy membership as $\dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})}$. Holding $w(\mathbf{x})$ fixed as $\mathbf{x}$ moves from left ($\underline{x} = \underline{y}$) to right($\overline{x} = \overline{y}$) this transformation maps linearly from 1 to 0. This corresponds to the case that if $\underline{x} = \underline{y}$ then no payoff in $\mathbf{y}$ is less than every payoff in $\mathbf{x}$ while $\mathbf{y}$ contains payoff values which exceed every possible payoff value in $\mathbf{x}$, so $\mathbf{x} \leq \mathbf{y}$ in a crisp sense. Notice that this membership value also corresponds to the proportion of the set $[\underline{y}, \underline{x}] \cup [\overline{x}, \overline{y}]$ which is greater than or equal to $\overline{x}$. Similarly, in case $\overline{x} = \overline{y}$ we get $\mathbf{y} \leq \mathbf{x}$ in a crisp sense. In a parallel fashion we define $\mathbf{x} \geq \mathbf{y}$ in terms of fuzzy membership as $\frac{\underline{x} - \underline{y}}{w(\mathbf{y}) - w(\mathbf{x})}$. This value, as above, corresponds to the proportion of the set $[\underline{y}, \underline{x}] \cup [\overline{x}, \overline{y}]$ which is less than or equal to $\underline{x}$.

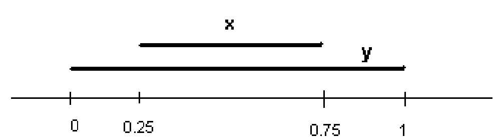

Figure 3: Nested intervals, $\mathbf{x} \leq \mathbf{y}$ with membership value $\frac{1}{2}$

We note that in case 3.2 if $\mathbf{x} \leq \mathbf{y}$ with fuzzy membership value α then, if a number is randomly chosen in the set $[\underline{y}, \underline{x}] \cup [\overline{x}, \overline{y}]$ there is a probability of α that the value is greater than $\overline{x}$. Hence if $\alpha > \frac{1}{2}$ there is more likelihood that a randomly chosen element in $\mathbf{y}$ is not less than $\underline{x}$. Using simple algebraic operations, it can be seen that the membership value for $\mathbf{y} \geq \mathbf{x}$ is one minus the membership value for $\mathbf{x} \leq \mathbf{y}$, and this dual relationship clearly holds for all of the previous crisp cases except when $\mathbf{x} = \mathbf{y}$.

Summarizing the above discussion, we define the fuzzy comparison operators $\preceq$ and $\succeq$ for two closed and bounded intervals as follow:

Definition 4: The binary fuzzy operator $\preceq$ of two intervals $\mathbf{x}$ and $\mathbf{y}$ returns a real number between 0 and 1 as follows:

$$\mathbf{x} \preceq \mathbf{y} = \begin{cases} 1 & \mathbf{x} = \mathbf{y};\ \text{or } \overline{x} \leq \underline{y}, \mathbf{x} \neq \mathbf{y};\ \text{or } \underline{x} < \underline{y} < \overline{x} < \overline{y} \\ 0 & \overline{y} \leq \underline{x}, \mathbf{x} \neq \mathbf{y};\ \text{or } \underline{y} < \underline{x} < \overline{y} < \overline{x} \\ \frac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} & \underline{y} \leq \underline{x} \leq \overline{x} \leq \overline{y}, w(\mathbf{y}) > 0, \mathbf{x} \neq \mathbf{y} \\ \frac{\underline{y} - \underline{x}}{w(\mathbf{x}) - w(\mathbf{y})} & \underline{x} \leq \underline{y} \leq \overline{y} \leq \overline{x}, w(\mathbf{x}) > 0, \mathbf{x} \neq \mathbf{y} \end{cases} \tag{3}$$

Definition 5: The binary fuzzy operator $\succeq$ of two intervals $\mathbf{x}$ and $\mathbf{y}$ is defined as: $\mathbf{x} \succeq \mathbf{y} = 1$ if $\mathbf{x} = \mathbf{y}$ and $\mathbf{x} \succeq \mathbf{y} = 1 - (\mathbf{x} \preceq \mathbf{y})$ otherwise , i. e.

$$\mathbf{x} \succeq \mathbf{y} = \begin{cases} 1 & \mathbf{x} = \mathbf{y};\ \text{or } \overline{y} \leq \underline{x}, \mathbf{x} \neq \mathbf{y} \\ 0 & \overline{x} \leq \underline{y}, \mathbf{x} \neq \mathbf{y};\ \text{or } \underline{x} < \underline{y} < \overline{x} < \overline{y};\ \text{or } \underline{y} < \underline{x} < \overline{y} < \overline{x} \\ \frac{\underline{x} - \underline{y}}{w(\mathbf{y}) - w(\mathbf{x})} & \underline{y} \leq \underline{x} \leq \overline{x} \leq \overline{y}, w(\mathbf{y}) > 0, \mathbf{x} \neq \mathbf{y} \\ \frac{\overline{x} - \overline{y}}{w(\mathbf{x}) - w(\mathbf{y})} & \underline{x} \leq \underline{y} \leq \overline{y} \leq \overline{x}, w(\mathbf{x}) > 0, \mathbf{x} \neq \mathbf{y} \end{cases} \tag{4}$$

The comparison of two intervals can now be classified as either crisp or fuzzy as described below.

Definition 6: If the value of $\mathbf{x} \preceq \mathbf{y}$ is exactly one or zero, then we say that $\mathbf{x}$ and $\mathbf{y}$ are *crisply comparable*. Otherwise, we say that they are *fuzzily comparable*.

3 Crisply Determined Interval Matrix Game

In this section, we extend the concept of classical strictly determined games to interval matrix games whose row and column entries are crisply comparable.

Definition 7: Let $\mathbf{G}$ be a $m \times n$ interval game matrix such that all intervals in the same row (or column) of $\mathbf{G}$ are crisply comparable. If there exists a $\mathbf{g}_{ij} \in \mathbf{G}$ such that $\mathbf{g}_{ij}$ is simultaneously crisply less than or equal to $\mathbf{g}_{ik}$ for all $k \in \{1, 2, \cdots, n\}$ and crisply greater than or equal to $\mathbf{g}_{lj}$ for all $l \in \{1, 2, \cdots, m\}$ then the interval $\mathbf{g}_{ij}$ is called a *saddle interval* of the game. An interval matrix game is *crisply determined* if it has a saddle interval.

By the definition above, to determine whether an interval game matrix is crisply determined, one needs only to do the following:

1. For each row ($1 \leq i \leq m$), find the entry $\mathbf{g}_{ij^*}$ that is crisply less than or equal to all other entries in the i^{th} row.
2. For each column ($1 \leq j \leq n$), find the entry $\mathbf{g}_{i^*j}$ that is crisply greater than or equal to all other entries in the j^{th} column.
3. Determine if there is an entry $\mathbf{g}_{i^*j^*}$ that is simultaneously the minimum of the i^{th} row and the maximum of the j^{th} column.
4. If any of the above values cannot be found the game is not crisply determined. Otherwise, it is a crisply determined interval matrix game.

Example 8: Examine the interval game matrix (2), we found that $\mathbf{g}_{14}, \mathbf{g}_{23}$, and $\mathbf{g}_{31}$ are the minimum of rows 1, 2, and 3, respectively. And, $\mathbf{g}_{21}, \mathbf{g}_{12}$, $\mathbf{g}_{23}$ and $\mathbf{g}_{34}$ are the maximum of columns 1, 2, 3, and 4, respectively. Furthermore, $\mathbf{g}_{23}$ is simultaneously the minimum of the 2^{nd} row and the maximum of the 3^{rd} column. Hence, $\mathbf{g}_{23} = [1, 3]$ is a saddle interval of the game matrix. This is a crisply determined interval matrix game.

Mimicking the optimal strategy for a classical strictly determined game, we have the optimum strategies for both R and C in a crisply determined interval matrix game defined as:

- R should choose any row containing a saddle interval, and
- C should choose any column containing a saddle interval.

Theorem 9: If an interval matrix game is crisply determined, its saddle intervals are identical.

Proof: Let $\mathbf{G}$ be a crisply determined interval game matrix, and $\mathbf{g}_{ij}$ and $\mathbf{g}_{lk}$ are saddle intervals. Then, $\mathbf{g}_{ij} \leq \mathbf{g}_{ik} \leq \mathbf{g}_{lk}$, and $\mathbf{g}_{ij} \geq \mathbf{g}_{lj} \geq \mathbf{g}_{lk}$. Hence, $\mathbf{g}_{ij} = \mathbf{g}_{lk}$.

As in the classical case, the knowledge of an opponent's move provides no advantage since the payoff is assumed to be uniformly distributed within a saddle interval in a strictly determined interval game.

Definition 10: The *value interval* of a strictly determined interval game is its saddle interval. A strictly determined interval game is *fair* if its saddle interval is symmetric respect to zero, *i. e.* in the form of $[-a, a]$ for $a \geq 0$.

From Example 8 we know that $\mathbf{g}_{23}$ is a saddle interval of the matrix game (2). However, the midpoint of $\mathbf{g}_{23}$ is 2. Hence, the game is unfair since the row player has an average advantage of 2.

4 Fuzzily Determined Interval Matrix Game

For a general interval game matrix, the crisp comparability may not be satisfied for all intervals in the same row (or column). We now define the fuzzy memberships of an interval $\mathbf{v}_i$ being a minimum and a maximum of an interval vector $\mathbf{V}$; and then we define the notion of a least and greatest interval in $\mathbf{V}$.

Definition 11: Let $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be an interval vector. The fuzzy membership of $\mathbf{v}_i$ being a least interval in $\mathbf{V}$ is defined as $\mu(\mathbf{v}_i) = \min_{1\leq j\leq n}\{\mathbf{v}_i \preceq \mathbf{v}_j\}$. And, a least interval of the vector $\mathbf{V}$ is defined as an interval whose μ value is largest, i.e. $\mathbf{v}_{i^*} = \max_{1\leq i\leq n} \mu(\mathbf{v}_i)$.

Likewise, the fuzzy membership of $\mathbf{v}_i$ being a maximum interval in $\mathbf{V}$ is $\nu(\mathbf{v}_i) = \min_{1\leq j\leq n}\{\mathbf{v}_i \succeq \mathbf{v}_j\}$. Similarly, a greatest interval of the vector $\mathbf{V}$ is $\mathbf{v}_{i^*} = \max_{1\leq i\leq n} \nu(\mathbf{v}_i)$.

Example 12: Find the least and the greatest intervals for the interval vector $\mathbf{V} = \{[2,5], [3,7], [4,5]\}$.

Solution: We notice that $\mathbf{v}_2$ and $\mathbf{v}_3$ are not crisply comparable. By Definition 11, we have $\mu([2,5]) = 1$, $\nu([2,5]) = 0$; $\mu([3,7]) = 0$, $\nu([3,7]) = \frac{2}{3}$; and $\mu([4,5]) = 0$, $\nu([4,5]) = \frac{1}{3}$. Hence, the least interval of the vector $\mathbf{V}$ is $\mathbf{v}_1 = [2,5]$ with membership 1; and the greatest interval of $\mathbf{V}$ is $\mathbf{v}_2 = [3,7]$ with membership $\frac{2}{3}$.

Notice, however, that unlike real valued games, the least and/or greatest interval of a vector is not necessarily unique. This can happen only when unequal intervals share the same midpoint, as the next example shows.

Example 13: Given the interval vector $\mathbf{V} = \{[2,5], [3,6], [4,5]\}$ we find that the least interval of the vector $\mathbf{V}$ is $\mathbf{v}_1 = [2,5]$ with membership 1. However, as $\nu([2,5]) = 0$, $\nu([3,6]) = \frac{1}{2}$, and $\nu([4,5]) = \frac{1}{2}$ each of $[3,6]$ and $[4,5]$ is a greatest interval with membership value $\frac{1}{2}$.

Definition 11 provides us a way to fuzzily determine least and greatest intervals for any interval vectors. We are now able to define fuzzily determined interval matrix game as follows:

Definition 14: Let $\mathbf{G}$ be an $m \times n$ interval game matrix. If there is a $\mathbf{g}_{ij} \in \mathbf{G}$ such that $\mathbf{g}_{ij}$ is simultaneously a least and a greatest interval for the i^{th} row and the j^{th} column of $\mathbf{G}$, respectively, then $\mathbf{G}$ is a *fuzzily determined interval game*. We also call such $\mathbf{g}_{ij}$ a *fuzzy saddle interval* of the game with its membership as $\min\{\mu(\mathbf{g}_{ij}), \nu(\mathbf{g}_{ij})\}$.

It is obvious that the crisply determined interval game defined in Definition 7 is just a special case of fuzzily determined interval game with 1 as its membership. The game value of a fuzzily determined interval game can be reasonably defined as its fuzzy saddle interval with respect to its membership.

For the convenience of computer implementations, we summarize our discussion as the algorithm below.

Algorithm 15:

1. Initialization:

(a) Input interval game matrix $\mathbf{G} = \{\mathbf{g}_{ij}\}_{m\times n}$

(b) Initialize `FuzzilyDetermined` as `false`

2. Calculation:

   (a) Evaluate $\mu(\mathbf{g}_{ij})$ and $\nu(\mathbf{g}_{ij})$ for all $i = 1$ to m and $j = 1$ to n

   (b) For each of $i = 1$ to m, find j^* such that $\mu(\mathbf{g}_{ij^*}) = \max_{1\leq j\leq n}\{\mu(\mathbf{g}_{ij})\}$. Note: j^* depends on i

   (c) For each of $j = 1$ to n, find i^* such that $\nu(\mathbf{g}_{i^*j}) = \max_{1\leq i\leq m}\{\nu(\mathbf{g}_{ij})\}$. Note: i^* depends on j

3. Checking: For each of $i = 1$ to m and corresponding j^*, check if $\mathbf{g}_{ij^*}$ is also a greatest interval for the j^* column. If so,

   (a) Update `FuzzilyDetermined` to `true`

   (b) Record $\mathbf{g}_{ij^*}$ as a fuzzy saddle interval with its membership $\min\{\mu(\mathbf{g}_{ij^*}), \nu(\mathbf{g}_{ij^*})\}$.

4. Finding results:

   (a) If `FuzzilyDetermined` is `false`, the interval game is not fuzzily determined

   (b) Otherwise, the interval game is fuzzily determined. And, return the fuzzy saddle interval that has the largest membership among all recorded fuzzy saddle intervals. Note: the game is crisply determined if the resulting membership is 1.

The concept of fuzzily determined interval game in Definition 14 can be further generalized. For each $\mathbf{g}_{ij} \in \mathbf{G}$, the membership of $\mathbf{g}_{ij}$ being simultaneously a least and a greatest interval for the ith row and the jth column of G can be defined as $\phi(\mathbf{g}_{ij}) = \min\{\mu(\mathbf{g}_{ij}), \nu(\mathbf{g}_{ij})\}$. The entries of $\mathbf{G}$ with the largest value of ϕ can be considered as fuzzy saddle intervals. Therefore, for any interval game matrix, one can find its fuzzy saddle intervals with the membership of the largest value of ϕ. However, it may not make any practical sense if the membership value is too small. A general study on undetermined interval games is needed.

5 Conclusion and future work

In this paper, we have introduced two-person zero-sum interval valued matrix games. By defining fuzzy binary comparison relations, we extended the strategies for classical strictly-determined matrix games into fuzzily determined interval matrix games. This extention provides a method of handling uncertainty in decision making modeled by matrix games.

This clearly does not cover all possible cases for an interval matrix game as in the classical case. Some interval matrix game may be neither crisply nor fuzzily determined. One approach that we are investigating for such non-determined games is a consideration of a combination of different strategies. The concept of an interval valued fuzzy game can be extended to multi-players.

Acknowledgment

This work is partially supported by NSF Grant CISE/CCR-0202042.

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