

# Estimating Uncertainties for Geophysical Tomography

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**Abstract.** We present statistical and interval techniques for evaluating the uncertainties associated with geophysical tomographic inversion problems, including estimation of data errors, model errors, and total solution uncertainties. These techniques are applied to the inversion of traveltimes data collected in a cross well seismic experiment. The inversion method uses the conjugate gradient technique, incorporating expert knowledge of data and model uncertainty to stabilize the solution. The technique produced smaller uncertainty than previous tomographic inversion of the data.

## 1. Introduction: Goals of Geophysics, and How Statistical and Interval Computations Can Help to Achieve These Goals

### 1.1. FORMULATION OF THE PROBLEM IN GEOPHYSICAL TERMS

#### 1.1.1. *In Geophysics, Indirect Measurements Are Necessary*

From under the earth, come minerals, water, and other commodities. From the earth, also comes destruction: earthquakes and volcanos. The main goals of geophysics are:

- to locate minerals (oil, gas, fresh and saline water, etc.);
- to locate and predict earthquakes (*seismology*), and to help design buildings that can withstand earthquakes (*earthquake engineering*);
- to predict what will happen if we place some substances under the earth: e.g., if we use water to release oil, if we dump nuclear or toxic wastes into cavities (*environmental engineering*).

In the majority of these problems, we cannot directly measure the quantity  $y$  in which we are interested. For example, the only way to directly measure the amount of oil in an area is to drill several wells, but drilling is a very expensive procedure, and the whole idea of geophysics is to predict the amount of oil without drilling in all possible places. Since we cannot measure  $y$  *directly*, a natural idea is to estimate  $y$  *indirectly*, i.e.:

- to estimate some other (easier to measure) physical quantities  $x_1, \dots, x_n$  that are related to  $y$ , and then
- to compute an estimate  $\tilde{y}$  for  $y$  based on the results  $\tilde{x}_1, \dots, \tilde{x}_n$  of measuring the quantities  $x_1, \dots, x_n$ .

For example, we measure the characteristics  $x_1, \dots, x_n$  of the sound wave that has passed through a region of interest, and we want to reconstruct the geophysical properties of this region.

### 1.1.2. *Indirect Geophysical Measurements Allow a Solution to a Tomographic Inverse Problem*

In geophysics, we usually know the equations that describe the propagation of the signal in the area. So, if we know the characteristics  $y_1, \dots, y_m$  of the medium (e.g., velocities of sound in the points through which the sound wave has passed), we can solve these equations and compute the resulting characteristics of the signal  $x_i = f_i(y_1, \dots, y_m)$ ,  $1 \leq i \leq n$  (here,  $f_i$  denotes the algorithm that computes  $x_i$  from  $y_j$ ). The problem of computing  $x_i$  from  $y_j$  is called a *direct* problem, because it is indeed the direct problem of science: we have some information about the object, and we want to use this knowledge to make predictions. In reality, we do not know the values  $y_i$ , we want to find them. So, we have to determine the values  $y_i$  from the system of equations  $x_i = f_i(y_1, \dots, y_m)$ . The problem of finding  $y_j$  from  $x_i$  is in some sense *inverse* to the problem of finding  $x_i$  from  $y_j$ , and it is, therefore, called the *inverse* problem (see, e.g., [45]).

A specific feature of inverse problems in geophysics is that we can measure the characteristics  $x_i$  on the surface, and we have to predict the characteristics  $y_j$  of the deep layers. Methods of solving such problems are called *tomography*. The most well-known example is *medical tomography*, where we determine the properties of the human body by measuring the waves that have passed through the body at different angles.

### 1.1.3. *In Geophysics, We Usually Cannot Uniquely Determine the Desired Parameters from Measurement Results*

In *medical* tomography, we can place the source and the receiver at arbitrary places outside the patient's body. By careful placement of sources and receivers, we look at 2-D slices of the patient that can then be collected to form a 3-D image. This enables us to uniquely determine the parameters of the body.

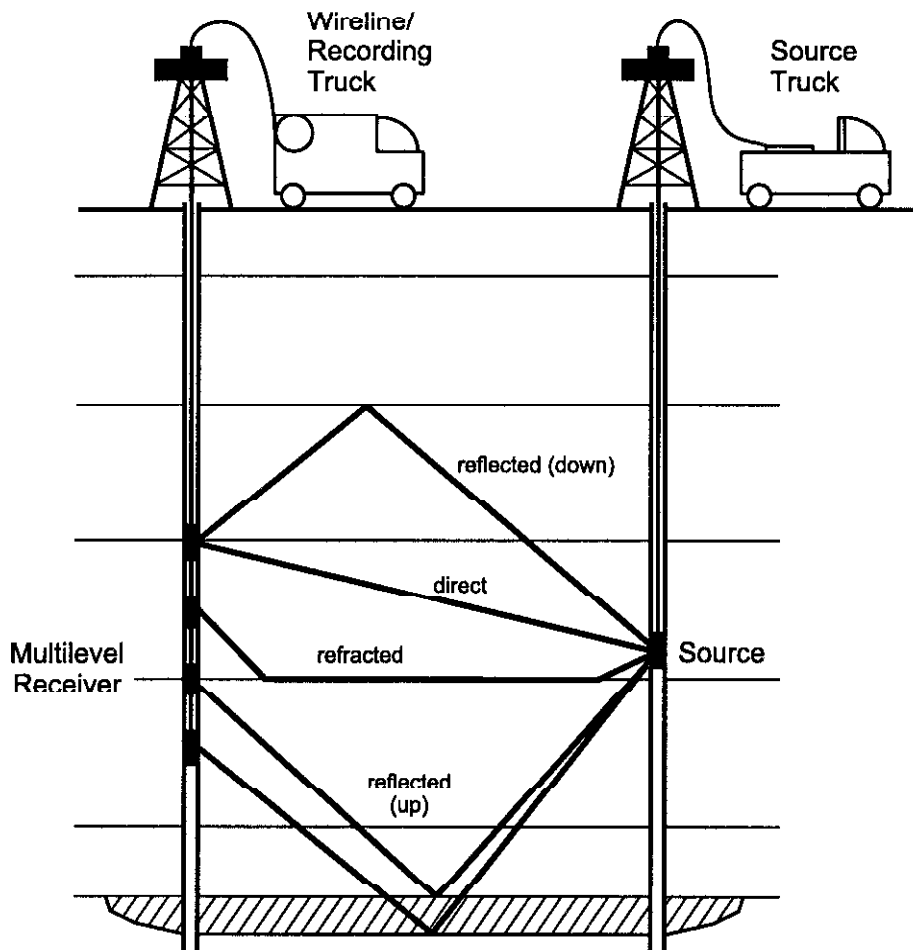


Figure 1. Illustration of how crosswell seismic traveltime data are collected (modified from [53]). The seismic tomography method is used to determine the velocity (or slowness) structure between the source well and receiver well.

In *geophysical* tomography, we can only place sources and receivers at the surface, or within shallow boreholes (Figure 1); hence, we do not get a  $360^\circ$  coverage. As a result, the total number  $n$  of parameters  $x_i$  that we can measure is often much smaller than the number  $m$  of parameters  $y_1, \dots, y_m$  that we want to know. So, we get an under-determined system of equations:  $n$  equations for  $m > n$  unknowns, and such systems usually have many different solutions. Although overdetermined systems ( $n > m$ ) are seldom encountered in geophysical applications, they too can be non-unique.

In view of this, the traditional goal of geophysics was to find *a* solution, i.e., a possible model that explained the existing measurement results. The question of the model's accuracy was not even raised because usually, other (radically

different) models could also explain the same data. This attitude was reflected in the following fact: When the authors of the first successful medical expert system MYCIN [12], [52] tried to apply their approach to geophysics, the main obstacle turned out to be the fact that in geophysics (due to impossibility to measure as many parameters as in medicine), experts have to be less cautious in making decisions. The level of uncertainty in a geophysicist's dream-come-true is an exploratory surgeon's nightmare (the fact that experts from different areas do think differently was confirmed by numerous psychological experiments; see, e.g., [58]).

#### 1.1.4. *In Many Geophysical Problems, It Is Now Important to Guarantee the Correctness of the Model and to Describe Its Accuracy*

Nowadays, geophysical measurements are getting better and better. New geophysical techniques and complicated computer processing algorithms enable us to predict subsurface properties to an extent that mining is not such a high-risk endeavor anymore. Hence, economic payout on success becomes far more predictable while errors have a very high cost. For example, if we want to decide whether a particular well is worth drilling, and the estimate for the amount of oil is  $\bar{y} = 100$  mln. tons, then before we start drilling, we would like to know whether the actual amount is, say,  $100 \pm 5$ , in which case, we should probably start drilling, or it is  $100 \pm 100$  (maybe 100, maybe 0, maybe 200), in which case we would rather undertake further (and more accurate) measurements.

Errors are especially intolerable in earthquake and environmental engineering, where a mistake can mean not only loss of money, but a disastrous loss of lives. For such problems, we want *guaranteed* estimates.

Therefore, we want to know not only *a* possible model, but we want to know *all* models that are consistent with the given observations and measurements.

The possibility for such a guaranteed estimate stems from the above-mentioned fact that in the last decades, the quality and quantity of geophysical measurements have drastically increased. The results of these measurements enabled the experts to understand geophysical processes much better. Therefore, now, experts can often conclude that out of all mathematically possible solutions  $y_1, \dots, y_m$  of the system  $x_i = f_i(y_1, \dots, y_m)$ , only one (or, at most, a few) make geophysical sense. In other words, in many reasonable cases, in addition to  $n$  equations, we have *expert knowledge* that further restricts the values of  $y_i$ .

This expert knowledge is usually represented in terms of intervals of possible values of either the quantities  $y_i$  themselves, or some characteristics  $z_k$  related to  $y_i$ :  $z_k = g_k(y_1, \dots, y_m)$ . The first case can be viewed as a particular case of the second one, with  $g_k(y_1, \dots, y_k, \dots, y_m) = y_k$ . As a result, the general type of the expert knowledge can be represented as  $g_k(y_1, \dots, y_m) \in [z_k^-, z_k^+]$ ,  $1 \leq k \leq e$ , where  $z_k^\pm$  are given estimates, and  $g_k$  are given functions.

*Comment.* Historically the first expert data was described and used in [3], where they stated that the spatial derivatives of geophysical parameters cannot exceed a certain value. Further use of expert estimation was pioneered in [33] and [55].

As a result, we arrive at the following *problem*.

### 1.1.5. Problem

Suppose that we have processed the measurement results  $x_1, \dots, x_n$  and expert estimates  $z_1^-, z_1^+, \dots, z_e^-, z_e^+$ , and we came up with a model  $y_1, \dots, y_n$  that is consistent both with these results and with the expert knowledge:

- Can we guarantee that this model is *the only possible one* (i.e., is it true that every other set of values  $y_i$  that is consistent with the observations and with the expert estimates is close to our estimates  $\tilde{y}_i$ )?
- If the model is unique, then what is the *accuracy* of the resulting indirect measurement (i.e., how different can the actual values  $y_i$  be from our estimates  $\tilde{y}_i$ )?

## 1.2. INPUT DATA FOR THE PROBLEM

### 1.2.1. Required Input Data

In order to formulate these problems in mathematical terms, we must describe their input, the available data. According to our description, this data must include:

- functions  $f_1, \dots, f_m$  that relate the desired values  $y_j$  with the measured quantities  $x_1, \dots, x_n$ ;
- *measurement results*  $\tilde{x}_1, \dots, \tilde{x}_n$ ;
- information on the *accuracy* of the measurement results, i.e., on the possible values of measurement error  $\Delta x_i = \tilde{x}_i - x_i$ .
- functions  $g_1, \dots, g_e$  that relate the desired values  $y_j$  with the quantities  $z_1, \dots, z_e$  estimated by the experts;
- *expert estimates*  $z_1^-, z_1^+, \dots, z_e^-, z_e^+$ ;
- information on the *accuracy* of expert estimates, i.e., on the possible values of measurement error  $\Delta z_k = \tilde{z}_k - z_k$ , where  $\tilde{z}_k = (z_k^- + z_k^+) / 2$ .

### 1.2.2. Traditional Data Processing Techniques Are Based on Gaussian Distributions

In traditional measurement theory [25], [49], [57], it is usually assumed that we know the probabilities of different values of measurement errors  $\Delta x_i = \tilde{x}_i - x_i$ , and moreover, that these probabilities are normally distributed. For such situations, there exist numerous methods that compute statistical characteristics of the resulting

error  $\Delta y = \bar{y} - y$  (see, e.g., [25], [49], [57]). Currently, these methods are the most frequently used in geophysics.

A serious problem with these methods is that they have been designed for processing *measurement results*, and in geophysical problems, we also have *expert estimates*, whose inaccuracy is described by intervals. So, in order to apply Gaussian techniques to expert estimates, it is usually assumed that the errors of these expert estimates are also distributed according to the Gaussian law. The standard deviation  $\sigma[z_k]$  of the corresponding Gaussian distribution is determined from the known fact that the actual values of the quantities  $z_k = g_k(y_1, \dots, y_m)$  belong to the expert's intervals  $\mathbf{z}_k = [z_k^-, z_k^+]$  in about 95% of the cases. So, we choose  $\sigma[z_k]$  in such a way that a distribution with a center in  $\bar{z}_k$  has a 95% probability to be inside this interval. For a Gaussian distribution, this probability is attained for  $2\sigma[z_k]$  deviations. Therefore, we determine  $\sigma[z_k]$  from the equation  $\mathbf{z}_k = [\bar{z}_k - 2\sigma[z_k], \bar{z}_k + 2\sigma[z_k]]$ ; so,  $\sigma_k^k = \Delta_k^{(z)} / 2$ , where we denoted  $\Delta_k^{(z)} = \bar{z}_k - z_k^- = z_k^+ - \bar{z}_k = (z_k^+ - z_k^-) / 2$ .

### 1.2.3. In Geophysics, Distributions Are Often Not Gaussian

The main fundamental motivation to use Gaussian distributions is that according to the central limit theorem, under reasonable assumptions, the distribution of the sum of several ( $N$ ) independent small random variables tends to the Gaussian distribution as  $N \rightarrow \infty$ . Therefore, if we eliminate major error components in the measurement error, the resulting error will be caused by the cumulative effect of many independent small components, and hence, its distribution will be close to Gaussian (see, e.g., [57]).

For example, in radioastronomical measurements, one of the major sources of error is the density of the water vapor in the troposphere. We can measure that density by an independent measurement device, pre-compute this error component, subtract it, and thus, eliminate this major source of error.

In some cases, error distribution in geophysical measurements is Gaussian (e.g., in gravity measurements). However, in other cases, the distribution is different. The reason why the above fundamental argument is not always applicable to geophysical measurements is that for these measurements, we know several major sources of error, but we cannot eliminate the corresponding error components because without the very drilling that we are trying to avoid, we cannot measure the corresponding error-inducing characteristics. As a result, the actual error distribution is often far from being Gaussian. For example, in some data, exponential distribution ([13], [28]) with probability density  $\text{const} \cdot \exp(-k|x|)$ , or a more general Weibull-type distribution with density  $\text{const} \cdot \exp(-k|x|^p)$  for some  $p > 0$  [26] are more adequate.

Part of the data comes from *experts*. The error distribution for expert estimates is also usually not Gaussian.

#### 1.2.4. In Geophysics, It Is Difficult to Determine the Probability Distribution

In traditional measurements, the probabilities of errors are usually obtained if we *calibrate* the measuring system, i.e.:

- we use the calibrated measuring system in conjunction with a much more accurate one (called a *standard*) in several measurements;
- for each measurement, we compute a difference  $e^{(k)} = \tilde{x}^{(k)} - x^{(k)}$  between the results of these two instruments, and use this difference as an estimate of the error of the measurement performed by the calibrated system;
- we reconstruct the error probability distribution from the recorded sample errors  $e^{(1)}, \dots, e^{(N)}$ .

In geophysical measurements, however, we are usually using top-of-the-line measuring instruments that have the highest accuracy possible, so, there is simply no “more accurate” measuring instrument that we can use for calibrating.

In those rare occasions when (for research purposes) error distributions have been estimated, it turned out that the probability distribution is not universal: it differs from site to site, so, we cannot assign a single probability distribution to a given measuring system. This fact is especially true for expert estimates, because the errors of expert’s estimates differ not only from site to site, but also from expert to expert.

#### 1.2.5. What Is Left Is Intervals

Since we do not know the probabilities of different measurement errors, the only information that we have about an error  $\Delta x_i = \tilde{x}_i - x_i$  is the guaranteed accuracy  $\Delta_i$  that is usually provided by the manufacturer of the measuring system, i.e., a guaranteed upper bound for  $\Delta x_i$ . For example, if  $\Delta_i = 0.1$ , this means that the measurement error cannot exceed 0.1. If our measurement result is  $\tilde{x}_i$ , then the possible values of  $x_i = \tilde{x}_i - \Delta x_i$  form an *interval*  $\mathbf{x}_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ .

Let us give examples of geophysical measurements in which intervals are a natural description of uncertainty:

- *Measuring velocity of sound.* When we have a well, we can extract samples from different depths and measure the correspondent velocities of sound. In velocity measurement, the major source of inaccuracy is that down in the well, where the sample was originally located, it was under a huge pressure. When we take the sample out, we release it from the pressure. As a result, the sample can expand. Also, when we lift the sample and thus decrease the pressure, gases that were filling the tiny cracks of the sample expand drastically. These expanding gases can widen the cracks and thus decrease the resulting density of the sample and increase the velocity of sound. Both effects lead to the fact that the measured velocity  $\tilde{v}$  can be smaller than the original velocity  $v$  of this sample. This difference can be up to 20% of  $\tilde{v}$ . So, the only thing that we can conclude

from the measurement result  $\bar{v}$  is that the actual velocity  $v$  belongs to an interval  $[\bar{v}, 1.2 \cdot \bar{v}]$ . The probability of different errors depends on the presence of gas, on the amount of mini-cracks, etc, and, as a result, these probabilities differ from site to site. If, instead of extracting the rocks, we place the velocity-measuring tool inside the well, we still get a different value, because, first, drilling damages rocks and distorts the resulting velocity, and second, the velocity of sound depends on the frequency, and the frequency used by the velocity-measuring tool differs from the velocity determined in the crosswell experiments.

- An important part of geophysical measurements is *measuring travelttime*  $t$ , i.e., time that is took for a signal to pass from the source (a natural earthquake or an artificial source) to the receiver. Here, the main problem is that the signals are usually rather weak. As a result, we may not be able to pick the first moment when the signal actually arrives (this first arrival will be drowned in noise), and we can erroneously take the arrival of the *second* maximum of the seismic wave as the measured value  $\tilde{t}$  of the travelttime. We could also start picking the wave before the second maximum. As a result, if the measured value is  $\tilde{t}$ , the only thing that we can guarantee about the actual value  $t$  is that  $t$  is in the interval  $[\tilde{t} - T, \tilde{t}]$ , where  $T$  is the period of the signal wave. Probabilities of different errors inside this interval depend on the seismic noise level and therefore, drastically differ from site to site.

If the source is not a periodic, but rather an *explosion-type* signal, then it can also happen that the receiver will skip the first received signal that arrives by the shortest possible path and pick only a later signal that may be the result of several reflections, and therefore, stronger than the original one. In this case, we also have an interval of possible values of  $t$ .

Because of these and other examples, geophysicists agree that an interval is “*the most natural assessment of uncertainty*” (see, e.g., [45], Section 3.01). However, this description of uncertainty is rarely used in geophysics, simply because statistical methods (based on Gaussian distribution) are well-developed, while methods of processing interval data are not so developed and, mainly, not so well-known. So, we arrive at the following natural idea:

#### 1.2.6. *Idea: Let Us Apply Interval Computations*

*Since in geophysical problems, the inaccuracy in input data is described by intervals, let us apply interval computations techniques (see, e.g., [29], [30], [41]) to geophysical data processing.*

#### 1.2.7. *Beyond Intervals*

Estimates that we get using interval computations can be viewed as *worst-case* error estimates. Usually, if we know probabilities of different distributions, we can get better estimates. In geophysical problems, we do not know the probabilities  $a$



*priori*, so, we cannot apply statistical methods with given probability distributions. However, for each site, we usually make lots of measurements (about 1000), and *after* we have performed these measurements, we can try to use these measurement results to get the *a posteriori* distributions, and then to use these distributions to get better (narrower) estimates.

### 1.3. THE STRUCTURE OF THE PAPER

In this paper, we will consider all three approaches: Gaussian (in Section 3), interval (in Section 4), and an attempt to go beyond intervals (Section 5). These approaches will be illustrated by a case study presented in Section 2. A brief description of another case study (earthquake engineering) is given in the Appendix.

A major part of this paper first appeared as a M.S. Thesis [26] defended by one of the authors; the reader is referred to this thesis for technical details and tables of data. This research was also presented at the 2nd Borehole Seismics Conference (Tohoku University, Sendai, Japan, November 1993) [14].

## 2. Case Study

### 2.1. THE CHOICE OF THE CASE STUDY: TRAVELTIME MEASUREMENTS IN A CROSS WELL SEISMIC EXPERIMENT

Methods described in this paper are very general, and can be applied to various geophysical problems. As a case study, we want to take the techniques that are the most useful in practice; so, let us analyze which of possible measurements are most useful.

Without going to huge expense, we can only measure characteristics on the surface or in the shallow wells. The parameters  $x_i$  that we can measure include density  $\rho$ , velocity of sound  $v$ , gravitational and electromagnetic fields, electric currents, etc. To get some information about the properties of the deep layers, we must measure the signals that are somehow influenced by these layers. Since these layers are usually pretty inactive (in the sense that they do not generate strong sound or electromagnetic waves), we must either wait until some strong wave passes through the layer, or artificially generate the wave that will go through that layer. As a result, we have two basic geophysical techniques:

- *passive* screening, when we wait for an earthquake and then measure the earthquake waves after they have passed through the layer that we analyze;
- *active* screening, when we *generate* the signal, and measure the result of its passing through the analyzed layer.

Passive methods often lead to better results because waves generated by earthquakes are much stronger than anything that we can artificially generate, and therefore, because of the huge signal-to-noise ratio, we can achieve a very detailed knowledge of the region of interest. Passive methods are, therefore, invaluable for

*fundamental* geophysics: i.e., in learning about the deep structures of the mantle, crust, etc. However, since the times and directions of earthquakes are unpredictable, we cannot rely on passive methods for *practical* (everyday) geophysical problems. For such problems, active methods are mainly used.

The active screening can be done using either acoustic or electromagnetic (EM) signals. Rocks are generally poor conductors of electromagnetic waves, and the high contrast in physical properties leads to extremely non-unique interpretations. On the other hand, rocks are known to be great conductors of sound. Because of that, in the majority of geophysical situations, the sound signal passing through the rocks carries more unambiguous information about the medium than the electromagnetic one. As a result, seismic (acoustic) screening is the major experimental technique (although EM measurements do provide us with important additional knowledge).

The received signal can be characterized by the dependency  $r(t)$  of its amplitude on time. For active screening, the resulting signal is usually weak (barely above the noise), so, the actual amplitude can be only measured with a very low accuracy; for such a measurement, to be within 10% of the actual amplitude might be considered perfect, so, from each measurement, we get one decimal digit. These uncertain measurements do not bring us much information about the rocks. Another physical characteristic of the process can, however, be easily measured with a much better accuracy: the *traveltime*, i.e., the time that it took for the first signal from the source reach the receiver. Time can be measured with an accuracy 1% and better, so, from each measurement, we get 2 or more decimal digits. The amount of information from measuring traveltimes is much larger than the amount of information from measuring amplitudes, therefore, commonly, only traveltimes are processed.

So, we arrive at the following experiment: we have two wells, and measure traveltimes between the source placed at different depth in the first well, and the receiver placed in the second well (Figure 1).

## 2.2. EXPERT KNOWLEDGE

Traveltimes depend on the velocity of sound in different areas. So, as expert knowledge, we took the expert estimates of the velocities that were based on the experts' knowledge of the particular area and of the general geophysical patterns.

First, for both wells, we have the velocities of sound of the rocks at the edges of the wells. For places in the close vicinity of the well, the density (and hence, the velocity of sound) is close to the values obtained from these direct measurements. To estimate the velocities in between the wells, we can use the known geophysical fact that the change in velocity cannot exceed a certain value  $C$ . As a result, the velocity in a point at distance  $r$  from the well must belong to the interval  $[v^- - Cr, v^+ + Cr]$ , where  $[v^-, v^+]$  is the interval obtained from measuring velocity at a corresponding point of the well.

In addition to these intervals, we consider the values obtained from more sophisticated geophysical knowledge. This knowledge is usually presented not in the form of a single set of intervals, but in the form of several possible sets of intervals for velocities that constitute several geophysically possible *models*. Some of these models may turn out to be inconsistent with the measurement results. So, we must try all the models, and pick those that are consistent.

### 2.3. CASE STUDY: GENERAL MATHEMATICAL DESCRIPTION OF THE EXPERIMENT

We measure traveltimes  $t_1, \dots, t_n$  (i.e.,  $x_i = t_i$ ), and we want to estimate the velocities. The actual velocity is continuously changing with spatial coordinates. We know, however (as we have just mentioned), that velocities in nearby points are close. Therefore, it makes sense to divide the entire area between the wells into several *blocks*, and assume that within a block, the velocity is constant. So, the unknowns here are the values of the velocities of sound  $v_1, \dots, v_m$  in different blocks.

We are measuring traveltime, i.e., the time interval between the moment when the signal was emitted and when the first signal was received. Therefore, we are actually measuring the time that it takes for the signal to follow the fastest path between the source and the receiver. We know exactly the locations of the source and of the receiver. If we knew the exact values of the velocity, we would be able to describe the exact fastest trajectory between the source and the receiver. Usually, expert estimates give more or less accurate values of the velocity, so, we can determine the paths more or less accurately. For the path that corresponds to  $i$ -th measurement, let  $l_{ij}$  denote the length of the part of this path that goes through  $j$ -th block (0 if the path does not go through block  $j$ ). Then, the total traveltime is  $t_i = \sum l_{ij} / v_j$ . If we know the paths  $l_{ij}$  and the times exactly, we get a linear system for the variables  $1 / v_j$ . Therefore, in geophysical tomography, these inverse values  $s_j = 1 / v_j$  (called *slownesses*) are usually taken as unknowns (so,  $y_j = s_j$ ). In terms of slownesses, the above equation takes the form  $t_i = \sum l_{ij} s_j$ . The expert knowledge consists of approximate values of  $s_j$  for  $1 \leq j \leq m$ .

### 2.4. THE DESCRIPTION OF THE ACTUAL MEASUREMENT SETUP

We used a data set of traveltimes collected from a seismic crosswell experiment called "MWX" run in an oil field near Rifle, Colorado [1]. Figure 1 illustrates the experimental setup. In the MWX study the two wells used were separated by 34 m. We concentrated on data collected between depths of 1800 and 2100 m in a region believed to be highly seismically non-uniform due to the presence of thinly layered rock and to the presence of natural gas. A 2.2 kHz source transmitted seismic signals at a rate of 1.5 pulses/m. The receiver was held stationary for each transit of the source and then was moved 1.5 m between transits.

### 3. Gaussian Approach

#### 3.1. MATHEMATICAL FORMULAS

In the Gaussian approach, we know the standard deviation  $\sigma_i$  of the measurement errors, and the standard deviations  $\sigma[z_k]$  of the expert errors. We assume that the errors are independent random variables with 0 average. For Gaussian distributions, the maximum likelihood method leads to the following formula for determining the unknowns  $y_j$ :

$$\sum_{i=1}^n \frac{(f_i(y_1, \dots, y_n) - \tilde{x}_i)^2}{\sigma_i^2} + \sum_{j=1}^e \frac{(g_k(y_1, \dots, y_n) - \tilde{z}_k)^2}{(\sigma[z_k])^2} \rightarrow \min_{y_1, \dots, y_m}. \quad (3.1)$$

In particular, for our case study, we get the formula

$$\sum_{i=1}^n \frac{(\sum l_{ij} s_j - t_i)^2}{\sigma_i^2} + \sum_{j=1}^n \frac{(s_k - \tilde{z}_k)^2}{(\sigma[z_k])^2} \rightarrow \min_{s_1, \dots, s_m}. \quad (3.2)$$

For our case study, we must minimize a quadratic function, so, in principle, we have a linear system. However, the number of equations and unknowns is about 1000, so, we cannot use the existing methods of solving linear systems. Instead, to solve these minimization problems, we used the conjugate gradient technique [50].

The problem is to estimate the accuracy of the resulting estimates. In general, formulas for the standard deviations and correlation matrix for  $s_j$  are well known in statistics [25], [57]; these methods, however, are based on the solution of the system of linear equations. We cannot directly apply these methods to our case, since the large size of the system precludes this type of solution.

#### 3.2. PREVIOUSLY USED METHODS FOR ESTIMATING UNCERTAINTY

Previously, the following methods have been used:

##### 3.2.1. Hit Count

The idea of this method is that blocks having more rays (or “hits”) pass through them are supposedly better sampled, and hence better resolved (i.e., the accuracy of  $s_j$  is better). This idea sounds reasonable; however, as we will demonstrate later in this paper, the “hit count” method does not provide a useful estimate of solution uncertainty.

##### 3.2.2. Monte-Carlo Approaches

These methods were first applied to geophysical data in [48].

In [31], Gaussian noise  $n_i$  (with 0 average and standard deviation  $\sigma_i$ ) is added to the measured values  $t_i$ , after which the same inversion technique is applied to the

same expert estimates  $\tilde{z}_k$  and to the simulated traveltimes  $t_i + n_i$ . As a result, we get the modified values  $y'_j$ . We repeat this experiment several ( $N$ ) times, and use the results  $y_1^{(1)}, \dots, y_j^{(N)}$  of this experiment to estimate the standard deviation of  $y_j$  as the mean square average

$$\sqrt{\frac{1}{N} \sum_{p=1}^N (y_j^{(p)} - y_j)^2}. \quad (3.3)$$

This method *underestimates* the errors in  $s_j$ , because it does not take into consideration that the expert estimates are also only approximately known.

In [32], simulated random errors  $n_k$  are added not to the traveltimes, but to the expert estimates  $\tilde{z}_k$ . The resulting error characteristics of  $y_j$  take into consideration inaccuracy of expert estimates, but not of the traveltimes and are, therefore, *underestimates*.

### 3.2.3. Bootstrap Approach

In the bootstrap approach [32], to estimate the error in  $y_j$ , we take several ( $N$ ; usually 100 or more) subsets of the original set of measurements results. For each subset  $p$ ,  $1 \leq p \leq N$ , we solve the inversion problem using only the data from this subset, find the values  $y_j^{(p)}$ , and then estimate  $\sigma[y_j]$  by using formula (3.3). In [46], the bootstrap approach is applied to earthquake location.

This method *underestimates* the errors in  $y_j$ , because it does not take into consideration that the expert estimates are also only approximately known.

*Comment.* The fact that all these methods underestimate the errors in  $y_j$  was first shown in [4], [5].

## 3.3. PROPOSED METHOD

We suggest the application of a Monte-Carlo approach in which we add Gaussian noise *both* to  $x_i$  and to  $z_k$  (i.e., in our case study, both to  $t_i$  and to  $\tilde{z}_k$ ). Then, we can use formula (3.3) to find the standard deviation of  $y_j$ .

This method is easily *parallelizable*, because we can run different simulations on different processors [37]. In [56], parallelization is described for a mesh architecture. In [9], [10], parallelization is described for a network of workstations.

## 3.4. APPLICATION TO CASE STUDY

We applied this method to MWX data with  $\sigma_t = 0.2$  msec, for four different models (sets of expert estimates): two isotropic and two anisotropic. Figures 2 and 3 show slownesses according to the anisotropic a priori models, along with the results obtained from inversion of a data set of perturbed traveltimes and a priori slownesses.

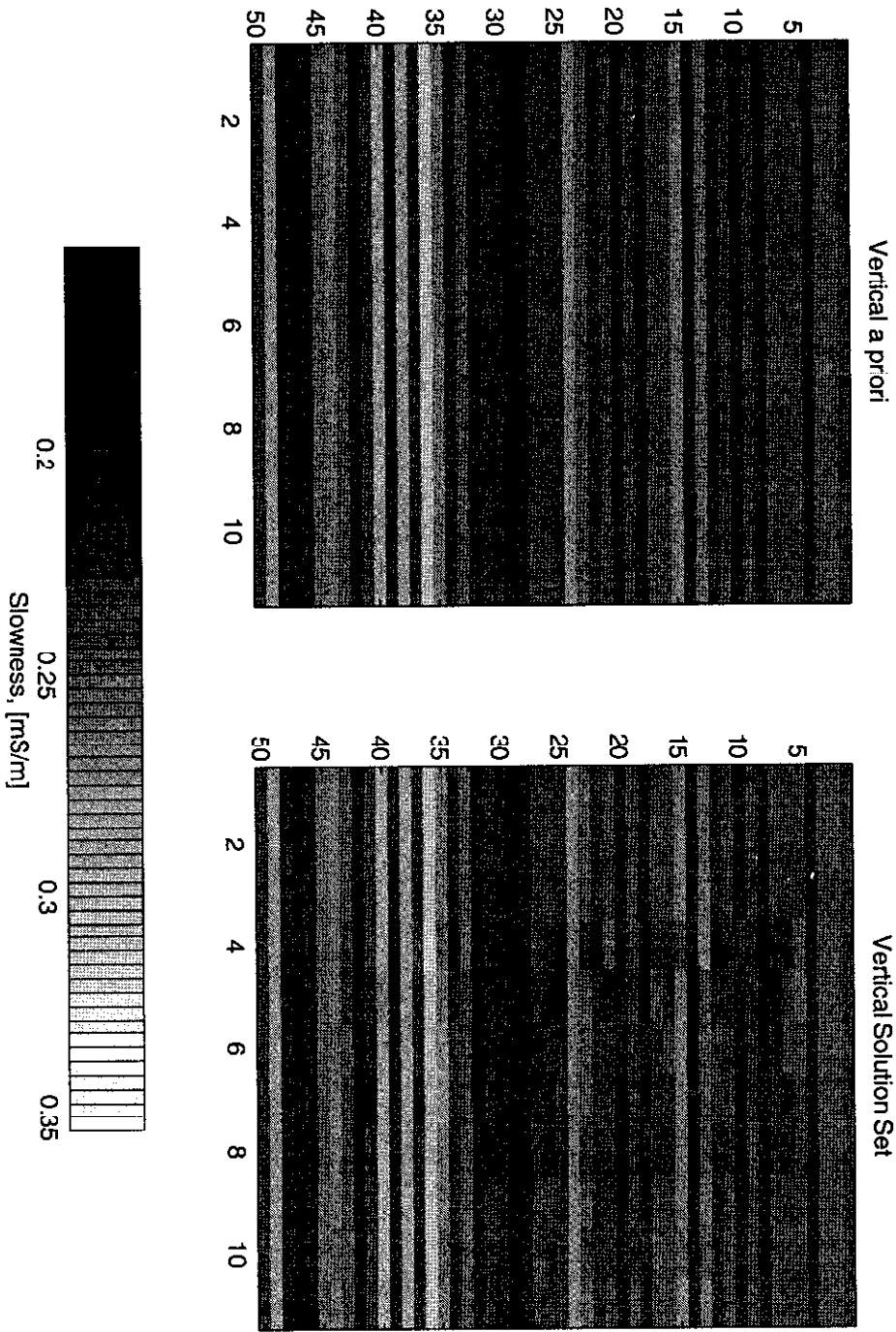


Figure 2. Vertical a priori slowness model (right) and vertical slownesses obtained from the inversion of perturbed traveltimes and a priori slowness data (left).

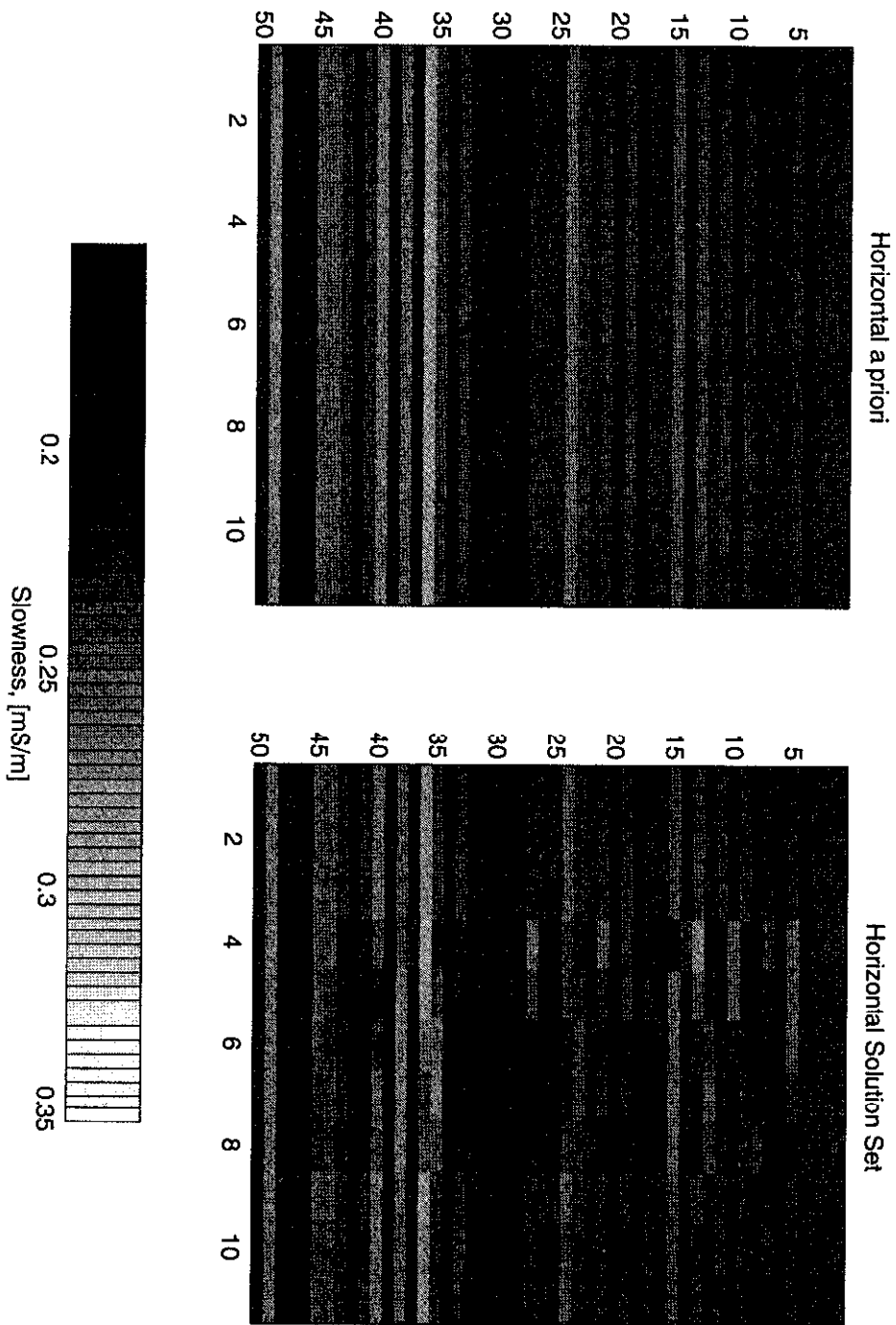


Figure 3. Horizontal a priori slowness model (right) and horizontal slownesses obtained from the inversion of perturbed traveltimes and a priori slowness data (left).

To find the appropriate number of Monte-Carlo iterations  $N$ , we repeated this inversion process for  $N = 1, 2, \dots$  (we actually did different iterations in parallel [42]). We found that after  $N = 7 - 10$ , these estimates practically do not change, which means that after 10 iterations, we get a correct estimate. The resulting standard deviations are shown in Figures 4 and 5. As expected, the velocity uncertainties are small close to the wells (where expert estimates were narrower) and larger farther from the wells. The figures also indicate that there are regions of large uncertainty; some of these regions appear to correspond with portions of the well where highly emergent seismic waves were observed Phillips [47]. When seismic waves are emergent, their exact arrivals in time are difficult to pick, leading to larger traveltimes uncertainties.

The number of “hits” per block is compared to horizontal and vertical total uncertainties in Figure 6. Note that hit density is not a reliable predictor of block slowness uncertainty.

## 4. Interval Approach

### 4.1. GENERAL CASE

In general, if we know the values  $\tilde{x}_i$ ,  $\Delta_i$ ,  $\tilde{z}_k$ , and  $\Delta_k^{(z)}$ , then the problem is, for every  $j$ , to find the set (interval) of possible values  $y_j$  for which for some  $y_1, \dots, y_m$ , the following inclusions hold:  $f_i(y_1, \dots, y_n) \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ ,  $g_k(y_1, \dots, y_n) \in [\tilde{z}_k - \Delta_k^{(z)}, \tilde{z}_k + \Delta_k^{(z)}]$ .

### 4.2. CASE STUDY

In particular, for our case study, if we can determine the intervals  $l_{ij}$  of possible values of path lengths, then, to find the intervals of possible values of slownesses, we must solve the following system of linear interval equations:  $\sum l_{ij}s_j = t_i$  and  $s_j \in s_j$ , where  $t_i = [\tilde{t}_i - \Delta_i, \tilde{t}_i + \Delta_i]$  and  $s_j = [\tilde{s}_j - \Delta_j^{(z)}, \tilde{s}_j + \Delta_j^{(z)}]$  (here, by a solution, we mean the tuple  $(s_1, \dots, s_m)$  for which  $s_j \in s_j$  for all  $j$ , and  $\sum l_{ij}s_j = t_i$  for some  $l_{ij} \in l_{ij}$ , and  $t_i \in t_i$ ). This problem can be solved, e.g., by one of the methods from [43].

Often, we can neglect the inaccuracy in  $l_{ij}$  and consider a *simplified* system: find  $s_j$  from the conditions that  $\sum l_{ij}s_j \in t_i$  and  $s_j \in s_j$ . This is a *linear programming* problem: to find the upper bound of the interval of possible values of  $s_j$ , we must solve the problem  $s_j \rightarrow \max$  under the above linear inequalities ( $s_j \rightarrow \min$  for the lower bound). Algorithms for solving linear programming problems are well known. This approach was used for our case study, with the resulting intervals for  $s_j$  3 to 30 times larger than in the Gaussian case.



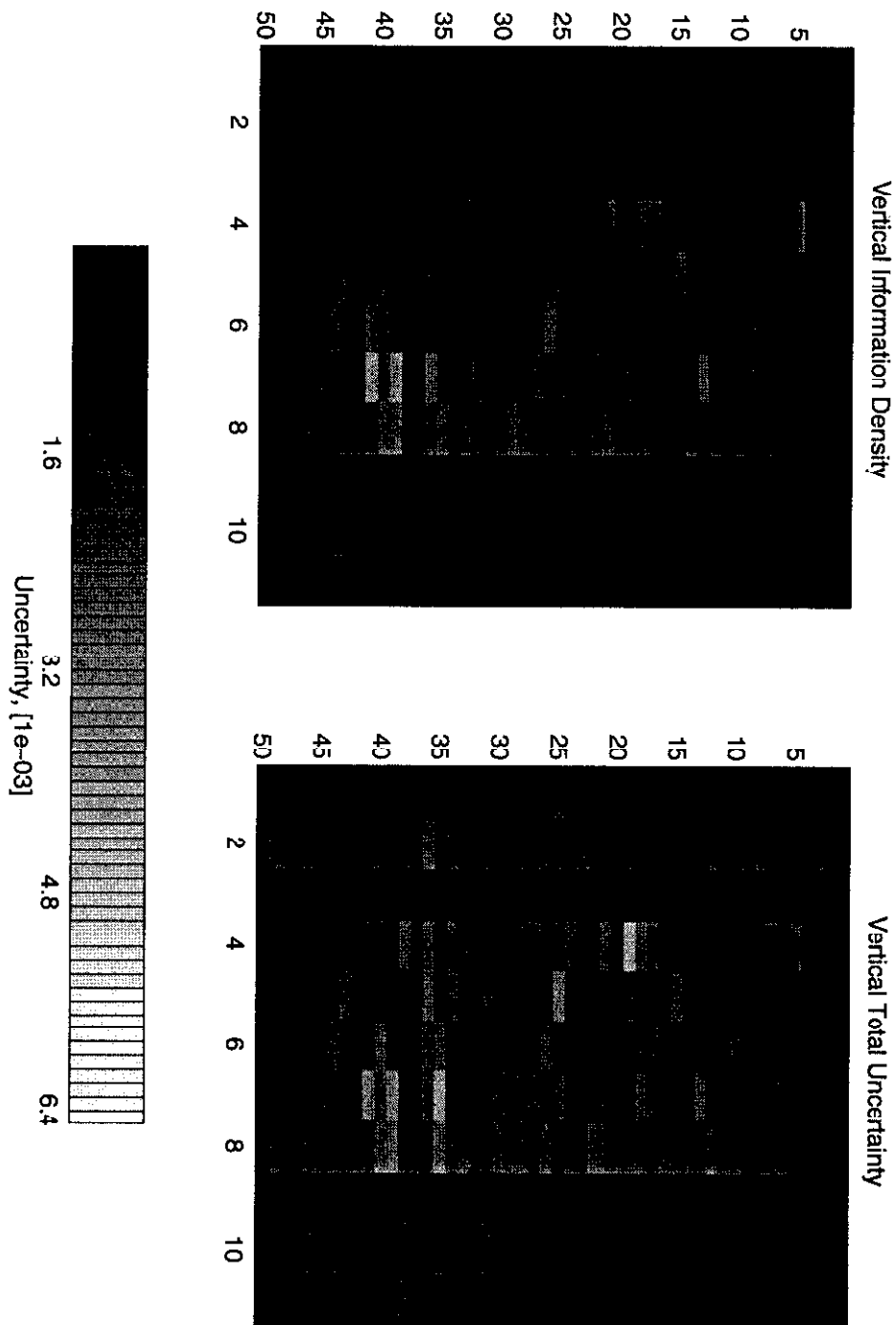


Figure 4. Vertical information density uncertainty (left) and total vertical slowness uncertainty (right) obtained from the inversion of 10 subsets of perturbed traveltimes and a priori slownesses.

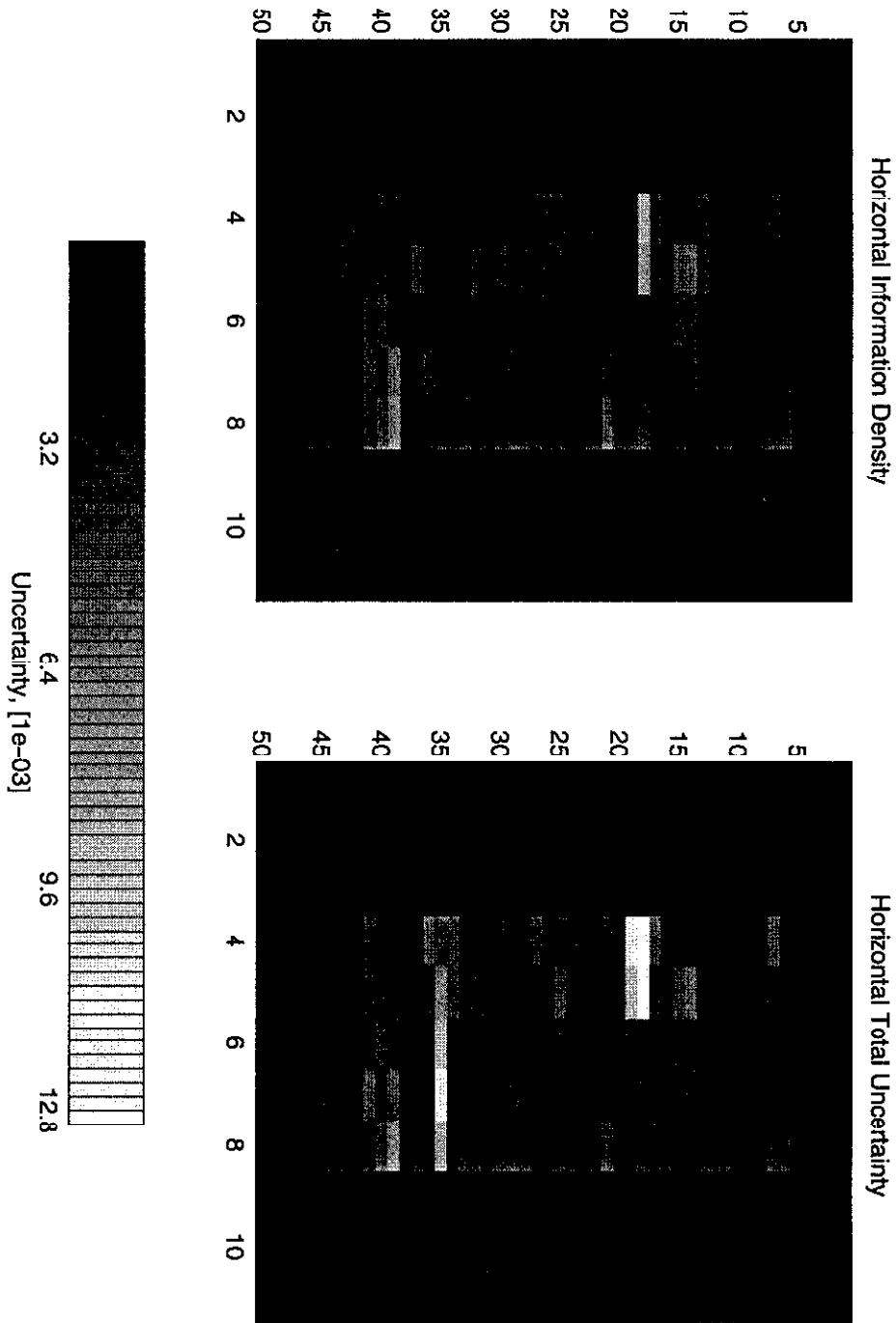


Figure 5. Horizontal information density uncertainty (left) and total horizontal slowness uncertainty (right) obtained from the inversion of 10 subsets of perturbed traveltimes and a priori slownesses.

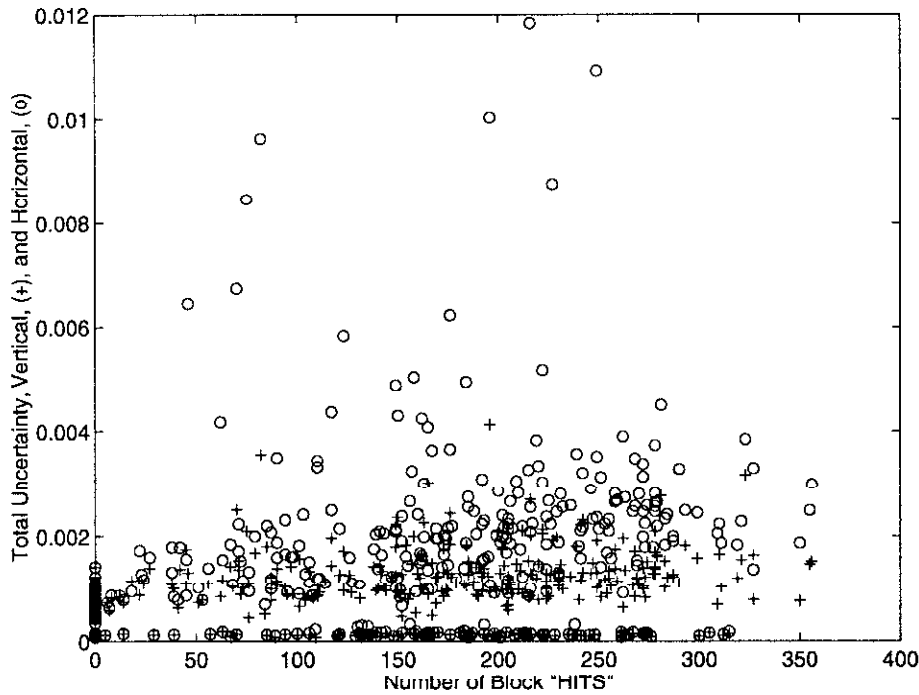


Figure 6. Number of hits per block versus the vertical and horizontal slowness uncertainty of the block.

### 4.3. COMBINATION OF INTERVAL AND STATISTICAL APPROACHES

#### 4.3.1. Checking Whether We Have Described All Possible Models

Interval computations can be used to check that we have found *all* possible solutions to an optimization problem (3.1).

#### 4.3.2. Estimating Accuracy of Statistical Models

**Necessity.** If we are not sure about the values  $\tilde{z}_k$ , and we feel that an *interval* of possible values of  $\tilde{z}_k$  will be a better description of our expert knowledge, then, we can use interval computations to find the *intervals* of possible values of  $y_j$ .

Another situation is when we know that the measurement error in  $x_i$  consists of two components: a (Gaussian) *random* component with a 0 average and known standard deviation, and a *systematic* component  $\Delta_{\text{sys}}x_i$ , about which we only know the interval  $[-\Delta_i^{\text{sys}}, \Delta_i^{\text{sys}}]$  of possible values. In this case, we can still apply formula (3.1), if instead of the actually measured value  $\tilde{x}_i$ , we take the difference  $\tilde{x}_i^{\text{corr}} = \tilde{x}_i - \Delta_{\text{sys}}x_i$ : this difference is distributed according to the Gaussian law. The only problem here is that we do not know the exact value of this corrected measurement result  $\tilde{x}_i^{\text{corr}}$ ; we only know that this corrected value belongs to the

interval  $\tilde{x}_i = [\tilde{x}_i - \Delta_i^{\text{sys}}, \tilde{x}_i + \Delta_i^{\text{sys}}]$ . So, to find the set of possible values of  $y_j$ , we must find *all* solutions of the system

$$\sum_{i=1}^n \frac{(f_i(y_1, \dots, y_n) - \tilde{x}_i^{\text{corr}})^2}{\sigma_i^2} + \sum_{j=1}^e \frac{(g_k(y_1, \dots, y_n) - \tilde{z}_k)^2}{(\sigma[z_k])^2} \rightarrow \min_{y_1, \dots, y_m}, \quad (3.1')$$

with  $\tilde{x}_i^{\text{corr}} \in \tilde{x}_i$ .

In our case study, we have used two methods to estimate the resulting intervals for  $y_j$ : naive interval computations and Monte-Carlo technique.

**Naive interval computations.** To estimate the interval of possible values of  $y_j$ , we can apply *naive interval computations*, i.e., we can repeat all computations of the inversion algorithm step-by-step, but with intervals instead of numbers [29], [30], [41].

**Monte-Carlo technique.** If we know that the intervals of possible values of  $\tilde{z}_k$  are narrow, and we can therefore approximate the desired dependency of  $s_j$  on  $\tilde{z}_k$  by a *linear* expression, then, we can use the following *Monte-Carlo* techniques described in [37], [39] (see [9], [10], [37], [56] for the parallelization):

Namely, if we have an algorithm  $F$  that transforms  $X_1, \dots, X_n$  into  $Y = F(X_1, \dots, X_n)$ , if we know the intervals  $\tilde{X}_i = [\tilde{X}_i - \Delta_i, \tilde{X}_i + \Delta_i]$ , and if the function  $F$  is linear in the neighborhood of  $\tilde{X}_i$ , then we can apply  $F$  to  $X_i^{(p)} = \tilde{X}_i + n_i^{(p)}$ ,  $1 \leq p \leq N$ , where  $n_i^{(p)}$  is distributed according to Cauchy law with a parameter  $\Delta_i$  (density  $\rho(n) = \text{const} \cdot (1 + (n / \Delta_i)^2)^{-1}$ ). We know that  $F$  is linear:  $F(\tilde{X}_1 + n_1, \dots, \tilde{X}_n + n_n) = \tilde{F} + F_{,1}n_1 + \dots + F_{,n}n_n$ , where  $\tilde{F} = F(\tilde{X}_1, \dots, \tilde{X}_n)$ , and  $F_{,i}$  is  $i$ -th partial derivative of  $F$  at a point  $(\tilde{X}_1, \dots, \tilde{X}_n)$ . Therefore, the resulting values  $Y^{(p)}$  are a linear combination of independent Cauchy-distributed random variables, and hence, the values  $Y^{(p)} - \tilde{F}$  are distributed according to Cauchy distribution with the parameter  $\Delta = |F_{,1}|\Delta_1 + \dots + |F_{,n}|\Delta_n$ . This is exactly the semi-width of the interval of possible values of  $Y$ . So, by applying statistical techniques to the sample  $Y^{(p)}$ , we can determine the desired value of  $\Delta$ .

## 5. Beyond Intervals

### 5.1. MAIN IDEA

Error estimates described by interval computations are an order of magnitude larger than Gaussian ones. If we knew the probability distribution of errors, we could have possibly ended up with smaller intervals. We do not know these distributions a priori, but after we have made lots of measurements, we can try to determine the distribution, and to apply the resulting distributions to get narrower intervals.

According to the experimental data analyzed in [44], the error distribution of the majority of measuring instruments can be described by a Weibull-type distribution with the probability density  $\text{const} \cdot \exp(-k|x|^p)$ . If we assume that both measurement errors and expert errors are distributed according to these laws (with parameters  $p$

and  $p^z$ ), then the maximum likelihood method leads to the following optimization problem:

$$\sum_{i=1}^n \frac{(f_i(y_1, \dots, y_n) - \bar{x}_i)^p}{\sigma_i^p} + \sum_{j=1}^e \frac{(g_k(y_1, \dots, y_n) - \bar{z}_k)^{p^z}}{(\sigma[z_k])^{p^z}} \rightarrow \min_{y_1, \dots, y_m} \quad (5.1)$$

*Comments.*

- This class of Weibull-type distributions can be also justified by the requirement that the corresponding maximum likelihood method lead to a scale-invariance formula, i.e., a formula whose minimum will be the same if we use a different unit to measure  $x_i$ ; see [11], [34], [36], [51].
- These distributions have been actively (and successfully) applied to geophysics; see, e.g., a monograph by Tarantola [54] and references therein. The existing applications of these distributions are based on the assumption that we already know the type of the distribution, i.e., that we at least know the value of the parameter  $p$ . In the existing applications, no algorithms for estimating  $p$  are used; instead, it is assumed that an *expert* estimates the value of  $p$  based on her expertise of the problem. Expert estimates are always subjective, and therefore, the resulting estimates for the desired geophysical parameters are not guaranteed. We proposed, instead, to estimate  $p$  from the same data as all other parameters of the unknown distribution (and to use general statistical techniques to estimate  $p$ ). As a result, in contrast to the existing applications, we can have different value of  $p$  for different portions of data (for example, in the above terms, we can have  $p \neq p^z$ ).

## 5.2. ALGORITHM AND ITS JUSTIFICATION

If we have a sample of values  $v_1, \dots, v_i, \dots, v_N$ , that is distributed according to a Weibull-type law with an unknown  $p$ , then we can estimate the parameter  $p$  from estimating the fourth moment of this distribution: namely, we can estimate the second and fourth moments as  $\mu_2 = \sigma[v]^2 = (1/N) \cdot \sum v_i^2$  and  $\mu_4 = (1/N) \cdot \sum v_i^4$ , estimate the excess  $\varepsilon = \mu_4 / \mu_2^2$ , and determine  $p$  from the equation  $\varepsilon = \Gamma(1/p) \cdot \Gamma(5/p) / (\Gamma(3/p))^2$ . A good approximation for  $p$  can be obtained by using the following formula [44], (5-27a):

$$p \approx \frac{1.46}{\ln(c - 2/9 - 10.7/\varepsilon^7) - 0.289}$$

Therefore, we can use the following iterative technique [34]:

- First, we find the Gaussian solution  $y_j$  using formula (3.1) (i.e., use  $p = 2$  as the initial approximation).
- After that, we compute the error values  $f_i(y_1, \dots, y_n) - \bar{x}_i$  and  $g_k(y_1, \dots, y_n) - \bar{z}_k$ . Applying the above-described technique to these values, we find the values  $p$  and  $p^z$  that are most adequate to describe these errors.

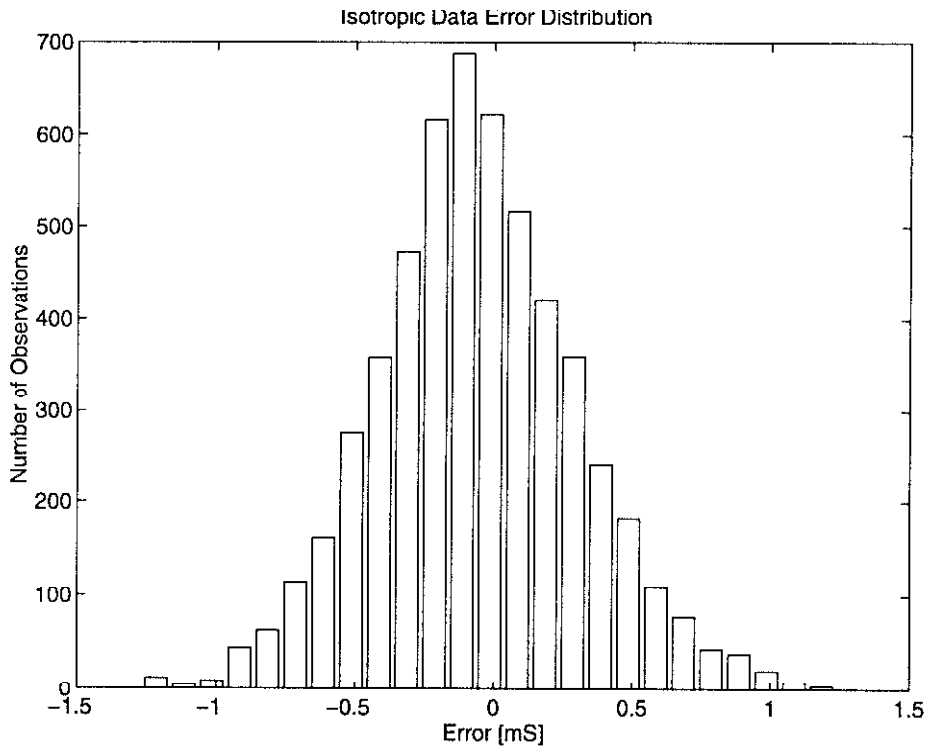


Figure 7. Distribution of data errors using an isotropic a priori slowness model.

- Then, we solve the system (5.1) with the newly found values  $p$  and  $p^z$ . (To solve this system, we used a slightly modified conjugate gradient technique of Scales et al. [50] that was originally developed for the case when, in our terms,  $p = p^z$ .) As a result, we get corrected values  $y_j$ . Then, we re-compute the errors using these corrected values of  $y_j$ , and compute  $p$  and  $p^z$  again. The process stops when the new values of  $p$  and  $p^z$  are practically equal to the ones from the previous iteration.

### 5.3. CASE STUDY

We applied this technique to our MWX data. For our examples, the second iteration lead to practically the same values of  $p$  and  $p^z$  as the first one; therefore, just one iteration was sufficient to determine  $p$  and  $p^z$ .

*Measurement errors* are shown in Figure 7. For these errors, the resulting values of  $p$  turned out to be between 1.8 and 1.9. This means that for measurement errors, the Gaussian model (with  $p = 2$ ) is a good approximation. On the other hand, since the values of  $p$  are different from 2 (and the difference is statistically significant), Gaussian methods do not give statistically correct estimates.

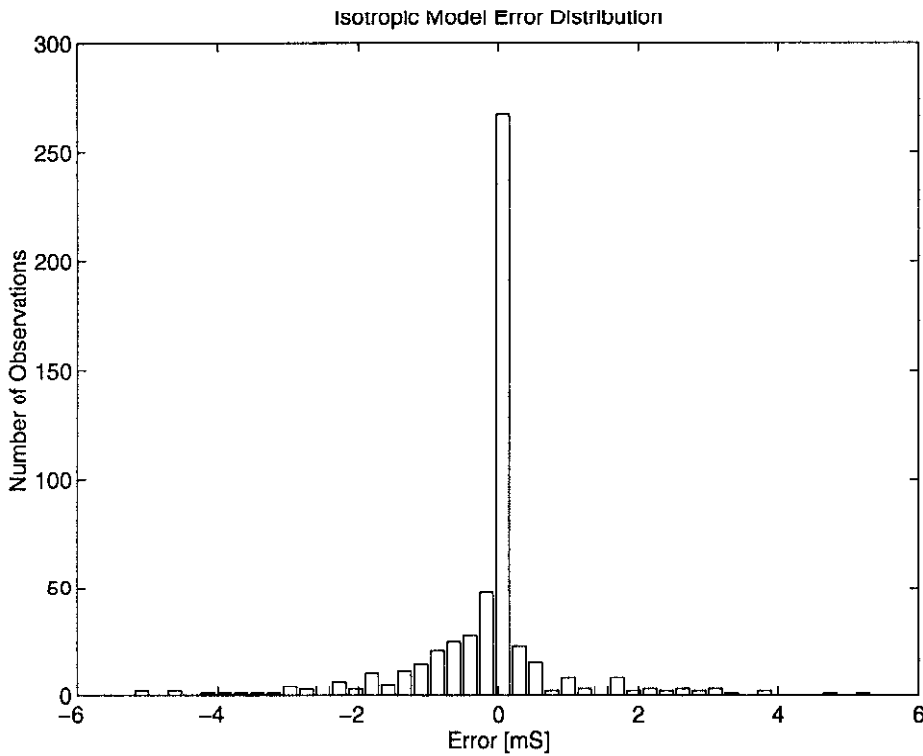


Figure 8. Distribution of model errors using an isotropic a priori slowness model.

*Expert errors* are shown on Figure 8. For these errors, the values  $p^z$  turned out to be between 0.7 and 1.1.

## 6. Conclusions

We have presented techniques to estimate uncertainties associated with seismic tomography problems. These techniques involves the explicit use of a priori (*expert*) model estimates and data uncertainties in the inversion process. The techniques can be applied to any geophysical problem for which a feasible inversion algorithm exists.

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## Appendix. Earthquake Engineering

**In earthquake engineering problems, linearization is impossible.** One of the main goals of civil engineering is to make sure that the designed structures do not collapse (or in general, do not fail). Collapsing is a very unstable process: small changes in the initial data may make a difference between stability and catastrophic failure. Therefore, linearization techniques, in which we neglect quadratic and higher order terms, are not always sufficient, and interval methods are needed (see, e.g., [35]). Another reason why we better not neglect any terms (even small ones) is that failures can be catastrophic, so we must have guaranteed estimates.

**Interval and generalized intervals.** In [35], an interval version of finite element analysis is used to describe stress and strain caused by loading. Generalized intervals (convex sets) are used to estimate stress concentration and predict collapse in [6], [7], [27]; in particular, these methods are used to predict natural frequencies of the structures [16]–[24], [40]. This is especially important for *earthquake engineering*, because during an earthquake, the major damage is caused by *resonance*, so we must be sure that the natural frequencies of the structure are different from the frequencies that we can expect during an earthquake.

**Using expert knowledge** ([15]). Earthquakes are extremely difficult to predict. Because of that, if we only use the equations and the measured data  $x_i$ , we get very wide intervals of possible magnitudes and frequencies  $y$ , intervals that are known to be much wider than the interval of actual values. So, if we use these wide intervals  $[y^-, y^+]$  to design a building that is guaranteed to withstand a typical earthquake, these buildings will be unnecessarily (and often unrealistically) expensive. To make the requirements more realistic, in addition to guaranteed measurement results, expert estimates are normally used. For every parameter  $x_i$  (e.g., for the frequency of earthquakes), estimates by different experts form a nested interval (sometimes, a nested set). In [15], for each expert, interval methods are used to compute the corresponding interval of  $y$  from this expert's estimates of  $x_i$ . To estimate the range of  $f$  on a box  $x_1 \times \dots \times x_n$ , the box is subdivided into several subboxes small enough that each subbox contains at most one local extremum point; for each subbox, the range is estimated. Then, the union of the estimates for subbox ranges is taken as an estimate for the total range. On each subbox, the maximum of  $f$  is attained either in the interior point (in which case this point is a local extremum of  $f$ ), or on one of the sides. For each side, we can repeat the similar procedure, and end up with a conclusion that the maximum is attained either in one of the corners, or in the local extremum point of one of the sides of smaller dimension. There are at most  $\approx 2^n$  vertices and local extremum points in each subbox, so, for small  $n$ , we can simply enumerate them all. The resulting intervals  $y$ , corresponding to different experts, are presented to the decision makers.

**Design in earthquake-resistance engineering.** During an earthquake, the major source of damage is a resonance: if the Earth vibrates with an eigen frequency of the construction, then even small vibrations, if applied for a sufficiently long time, will cause serious damage to the building. So, one way to prevent the earthquake damage is to change the eigen frequencies of a building so as to make them equal to the least powerful earthquake modes. Eigen frequencies are eigenvalues of a matrix  $A$  that describes the building's reaction to external forces. In order to change  $A$ , we may use springs to connect the building with additional objects. As a result of this connection, the matrix changes to  $A + c_1A_1 + \dots + c_nA_n$ , where  $A_i$  are matrices of these additional objects, and  $c_i$  are spring coefficients. In this case, the problem is: given the matrices  $A$  and  $A_i$ , to find the coefficients  $c_i$  for which the resulting matrix has the given eigenvalues. This problem is called an *inverse matrix eigenvalue*

*problem.* There are other modifications of this problem, in which we can change the characteristics  $A_i$  of the additional bodies. It is vitally important to be sure that the eigen frequencies are in the prescribed limits; therefore, it is reasonable to apply interval methods that give guaranteed bounds for the results. Interval methods for solving an inverse matrix eigenvalue problem have been proposed in [2] (actually, an interval version of the Newton's method is used to solve the corresponding system of equations).

**Dynamic control** ([38]). If it is impossible (or too expensive) to design an earthquake-safe building, in which all eigen frequencies are different from the earthquake's ones, then we can at least provide this building with a dynamic system that would change the building's eigen frequencies as soon as it becomes necessary. For that, we must constantly measure the displacements of different parts of the building, and estimate the eigenvalues based on the results of these measurements. In [38], interval methods are used for processing the measurements results.