INTERPOLATION OF AN INTERVAL-VALUED FUNCTION FOR ARBITRARILY DISTRIBUTED NODES

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1. Introduction and survey

The following problem is repeatedly posed in practice:
Given a set of N points $(x_i,y_i) \in \mathbb{R}^2$, and intervals $I_i = [u_i,o_i]$, $i=1(1)N$, such that for a function $f(x,y)$ to be interpolated the function values $f_i = f(x_i,y_i)$ satisfy $f_i \in [u_i,o_i]$, $i = 1(1)N$. The nodes (x_i,y_i) are assumed to be not collinear.

In order to construct an interpolating function of $f(x,y)$ the convex hull of the nodes (x_i,y_i) is triangulated, and then in every triangle a local polynomial $p_j(x,y)$ is defined. These local polynomials are now composed to give a continuous or even smooth differentiable global spline $S(f,x,y)$ such that $S(f,x_i,y_i) \in [u_i,o_i]$, $i=1(1)N$.

In order to obtain an applicable program for general $f(x,y)$ and $(x_i,y_i)_{i=1}^{N}$ there must be usually a priori more coefficients in $S(f,x,y)$ than conditions to define $S(f,x,y)$ uniquely. Hence a global condition is added, by which a suitable functional $\hat{N}f$ has to be minimized.

Finally the function $S(f,x,y)$ is represented and plotted by a set of curves of constant height.
The program is available in the Computer Center of the Technical University Aachen. Interested people should contact the second author.

2. The triangulation

The first problem is to obtain a suitable triangulation of the convex hull Ω of the nodes $(x_i,y_i),\dots,(x_N,y_N)$. There is first a theoretical point of view concerning the questions of unicity and a smallest number of triangles. Unicity is of course not given since the most trivial counterexample is the quadrilateral which is triangulated differently by the two diagonals. The smallest number of triangles depends on the distribution of the nodes. In Fig. 1 we have two examples with 11 nodes, but the number of of triangles is 9 in the first case, and 16 in the second case. If the boundary triangles are omitted (leaving the convexity!) then we have still 12 triangles.

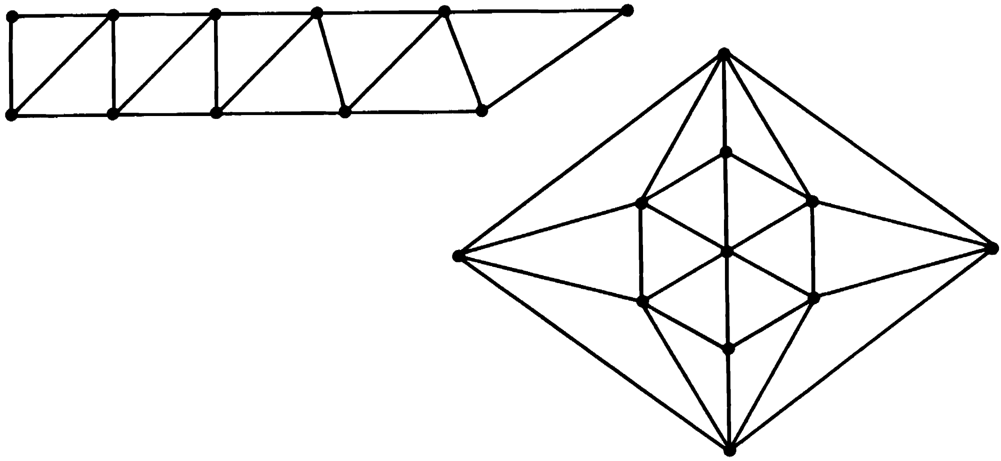

Fig. 1

The second point of view concerning the triangulation is a more practical one. It should be avoided (also with respect to error estimates as in FEM techniques) that very thin triangles similar to needles occur.

The triangulation is generated in two steps. At the beginning an arbitrary triangulation is generated, and this is improved several times. The initial triangulation is a more technical problem while the process of improving this initial triangulation is mathematically more interesting. In order to facilitate the computations we refer always to the standard triangle D_0 with nodes $(0,0)$, $(1,0)$, and $(0,1)$. As well known (e.g. from FEM-techniques) in Fig. 2 the right one of the two triangulations is to be preferred.

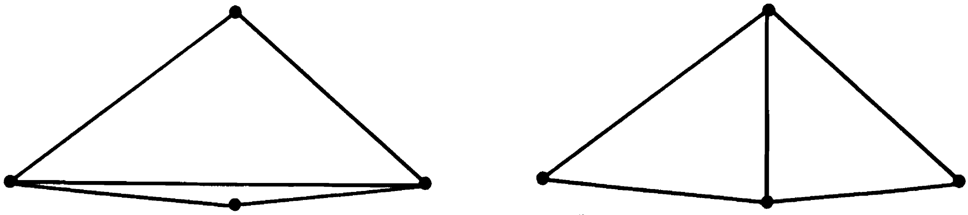

Fig. 2

So we improve the triangulation in such a way that

- the smallest angle φ (or $\sin\varphi$ resp.) of the set of triangles becomes as large as possible.

Alternatively we consider the area C_i of the osculating interior circle of a triangle with area Δ_i, and calculate the ratio $R_i := C_i / \Delta_i$. Then we obtain an equivalent but in view of programming more suitable criterion

- the minimum R of the ratios R_i of the set of triangles becomes as large as possible.

For details of the program see [2] .

3. The interpolation

Once the triangulation is performed we construct a polynomial of a given degree on each triangle. These are then put together to a spline $S(f,x,y)$ such that the resulting "surface" of the spline $S(f,x,y)$ passes through the given intervals $[u_i,o_i]$, and in addition is as smooth as possible. So we have a certain number of polynomial coefficients in each triangle which are related by transient conditions at the boundaries of adjacent triangles, and by interpolation conditions. We cannot expect that the number of coefficients equals the number of conditions. Hence the only way to circumvent this is to have more polynomial coefficients than necessary. To this end we use a result of Zenisek [3] : The wanted spline belongs to the class $C^m(\Omega)$ for any arbitrary triangulation if the partial derivatives of degree $\leq 2m$ are used at each node. In this case the polynomials have degree at least $4m+1$.
Hence for everyone of the N nodes we need at least

$$P := (2m+1)\,(m+1)$$

parameters in order to determine the

$$K := (2m+1)\,(4m+3)$$

coefficients of the M polynomials in Ω having degree

$$G := 4m+1 \quad .$$

The practically most interesting case is $m = 1$.

We now choose a suitable functional which is minimised in order to determine those coefficients which are not yet determined by corresponding conditions. This functional is

$$\hat{N}f := \sqrt{\sum_{j=1}^{M} \sum_{|\underline{s}|=o}^{m+1} \hat{a}_{j\underline{s}} \int_{D_j} [D^{\underline{s}} f(x,y)]^2 \, dx\, dy}$$

where M is the number of triangles, and $\underline{s}$ is a multiindex. The coefficients $\hat{a}_{j\underline{s}}$ allow a local weigthing of certain derivatives of f . $\hat{N}f$ gives a seminorm of f , and if $\hat{a}_{j\underline{s}} \neq 0$ it is even a norm. Hence we require that

$$\hat{N}f \overset{!}{=} \mathrm{Min}$$

for the set of interpolating splines still containing a lot of undetermined parameters. This leads to a semidefinite quadratic form satisfaying

$$(\hat{N}p)^2 = \underline{c}^T \, \underline{\underline{M}} \, \underline{c}$$

with a $Mk \times Mk$-matrix $\underline{\underline{M}}$, and the vector $\underline{c}$ of coefficients of the polynomials with restrictions $u_i \leq S(f,x_i,y_i) \leq o_i$, $i=1(1)N$. Now it is heuristically clear how to proceed, however the detailed discussion, and the proofs are far beyond the scope of this paper.

The fundamentals for the one-dimensional case are to be found in [1] . The extension of these results to the two-dimensional case is theoretically straight-forward but it is a very tedious and complex task to perform this extension.

The program [2] performs the minimisation of $\hat{N}$ subject to the restrictions

$u_i \leq S(f,x_i,y_i) \leq o_i$ by use of the steepest-descent method.

4. Representation of the results

Once the interpolating spline $S(f,x,y)$ passing through the intervals $[u_i,o_i]$, $i=1(1)N$, is computed, we must look for a suitable way to organize the output. Usually the user is not interested in the list of coefficients of the polynomials which determine the spline. He usually likes to see an global representation of $S(f,x,y)$ which gives the really needed information. This is in most cases attained by a plotted picture of $S(f,x,y)$ such as a 3-D-plot. However, in view of measuring a representation by lines of equal height is often preferred. This is also done in [2] . The main tools are successive refinement of the triangles, and determination of zeros on their boundaries.

5. Numerical examples

Finally we present two examples. The first one is based on the function plotted in Fig. 3 . The position of the local maximum is at the point (0.25, 0.5) . Fig. 4 shows the triangulation with 15 nodes. The derivatives of equal order in the functional $\hat{N}f$ are weighted uniquely so that the $\hat{a}_{j\underline{s}} = \hat{a}_j$ does no longer depend on $\underline{s}$. For $m=1$ we choose $\hat{a}_0 = \hat{a}_1 = 0$, $\hat{a}_2 = 1$, and obtain the results of Fig. 5 , where $|o_i - u_i| \leq 0.2$.

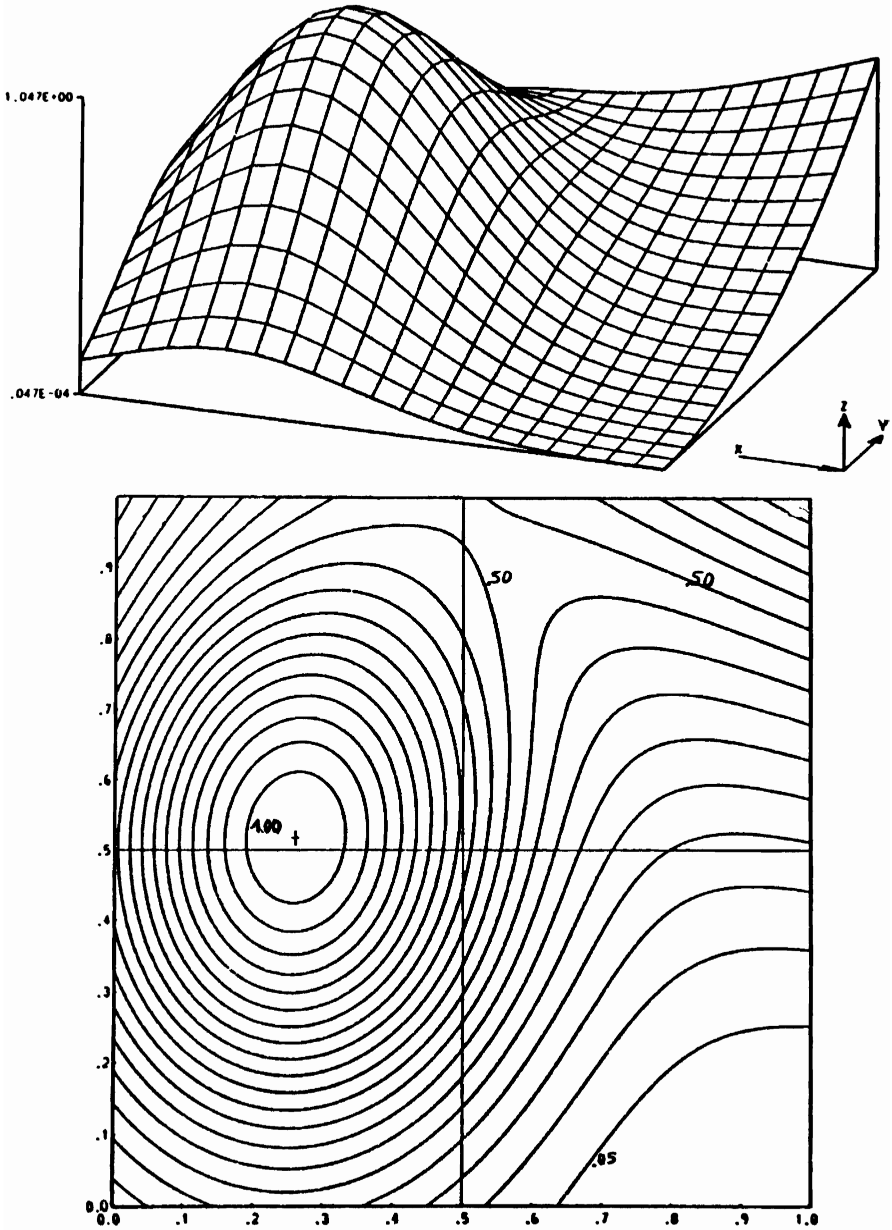

Fig. 3

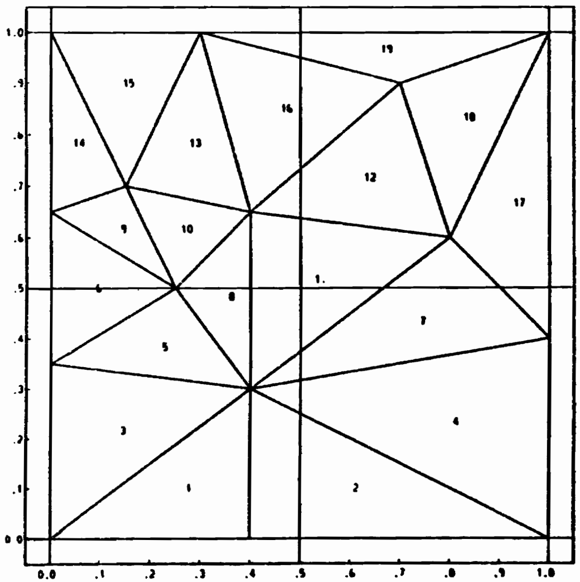

Fig. 4

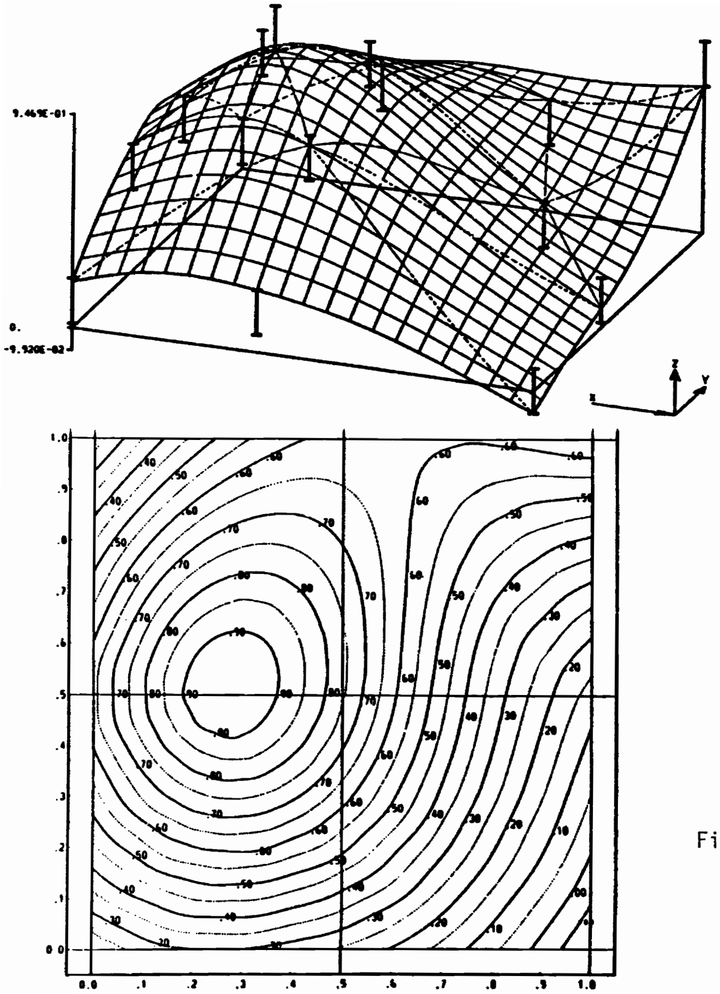

Fig. 5

The second example concerns a peaked function. This is approximately plotted in Fig. 6 using the program with $\hat{a}_0 = \hat{a}_2 = 0$, $\hat{a}_1 = 1$, and $u_i = o_i$, $i=1(1)N$ 15 .

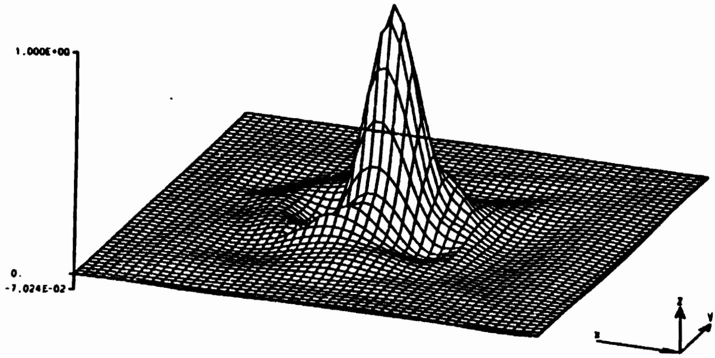

Fig. 6

The triangulation is as in Fig. 7 .

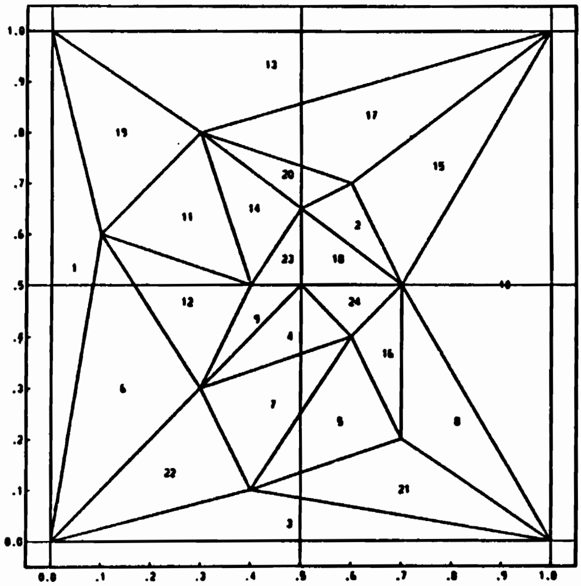

Fig. 7

In Fig. 8 and Fig. 9 we have the result for $\hat{a}_0 = \hat{a}_2 = 0$, $\hat{a}_1 = 1$ and $|o_i - u_i| < 0.2$.

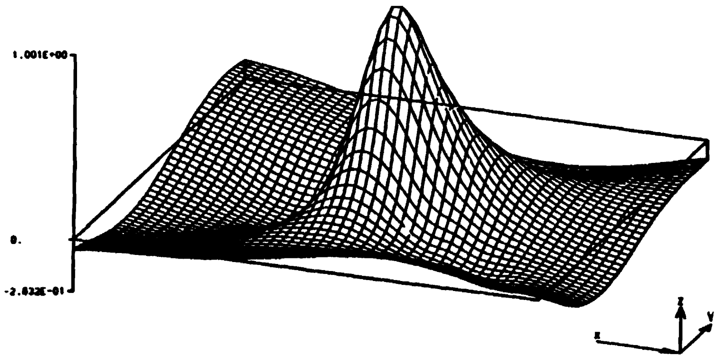

Fig. 8

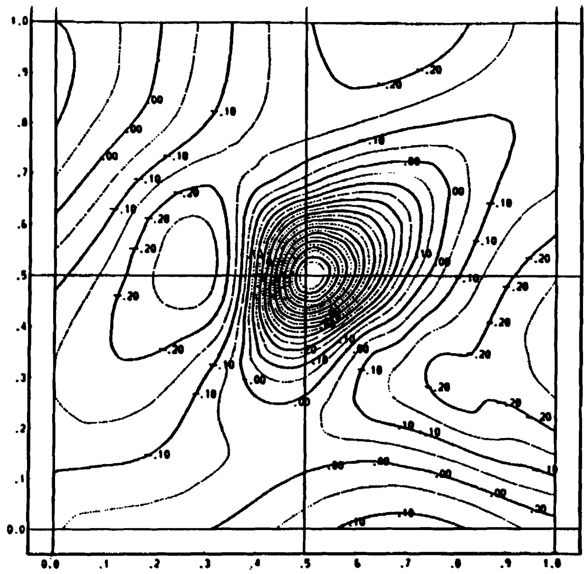

Fig. 9

We can say that this is a very "stiff" problem. The errors especially near the boundaries are not so small as one would like to see. In this case we get better results if the triangulation is refined (which causes a big additional amount of CPU-time) or by choosing a more appropriate (exponential or rational) spline.

References

[1] Groß, S. Problemorientierbare Interpolationsmethoden bei interwallwertigen Daten.
Thesis, Aachen 1980

[2] an May, D. Programm zur Erstellung eines Höhenlinienbildes einer Funktion, die durch einzelne, ungenaue Funktionswerte gegeben ist.
Computing Center of the Technical University Aachen, Seffenter Weg 23, 5100 Aachen

[3] Zenisek, A. A general theorem on triangular Finite $C^{(m)}$-Elements.
RAIRO 8 (1974), 119 - 127