Acceptable Solutions of Linear Interval Integral Equations

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I. Acceptable Solutions

Let us consider an integral equation of the form

$$\int_0^1 k(s,t)x(t)\,dt = y(s) \tag{1}$$

where $x \in X$, $y \in Y$, $k \in K$. We assume that X and Y are linear spaces of real functions on $[0,1]$ and K is a suitable collection of real functions on $[0,1]^2$.

Using the linear integral operator

$$A\colon X \to Y,\ Au = v,\quad v(s) := \int_0^1 k(s,t)u(t)\,dt$$

we can rewrite the equation (1) in the form

$$Ax = y\ . \tag{2}$$

Suppose we have calculated an $\bar{x} \in X$ with $A\bar{x} \approx y$, that is, $\bar{x}$ is an approximate solution of the given problem. Now one often says:

$\bar{x}$ is <u>acceptable</u> iff $\bar{x}$ is an exact solution of a slightly disturbed problem.

This notion is common, but it is meaningless as long as "slightly disturbed" is not defined. To be precise, we have to use some kind of tolerance regions, for instance intervals. We order the function spaces X and Y pointwise. The set $L(X,Y)$ of all linear operators $X \to Y$ can be ordered by

$$S \le T :\Longleftrightarrow Sx \le Tx \quad \text{for all } x \in X,\ x \ge 0\ .$$

Let LI be the set of all linear integral operators $X \to Y$. Of course the structure of LI depends on which linear operators are "integral". We defer the definition of "integral" until later. Nevertheless the order in $L(X,Y)$ induces an order in the subset LI. The order relations in X, Y, $L(X,Y)$ and LI enable us to use intervals. If H is an ordered set and $a,b \in H$ we define

$$[a,b]_H := \{h \mid h \in H,\ a \leq h \leq b\}.$$

We may drop the subscript of an interval, if the underlying ordered set is evident from the context.

For $\alpha \in LI$ with $\alpha \geq 0$ and $\eta \in Y$ with $\eta \geq 0$ it makes sense to define

$\overline{x} \in X$ is an acceptable approximate solution of problem (2) with respect to the tolerances α and η iff

$$\exists\ \overline{A} \in [A-\alpha, A+\alpha]_{LI},\quad \overline{y} \in [y-\eta, y+\eta]:\ \overline{A}\overline{x} = \overline{y}\ . \tag{3}$$

A quite different way to handle tolerances in problem (2) is the Interval Analysis approach.

For $\alpha \in LI$ with $\alpha \geq 0$ and $\eta \in Y$ with $\eta \geq 0$ we consider the following collection of equations:

$$\{\overline{A}x = \overline{y} \mid \overline{A} \in [A-\alpha, A+\alpha]_{LI},\quad \overline{y} \in [y-\eta, y+\eta]\}\ .$$

This collection may be written symbolically as "linear interval integral equation" in the form

$$[A-\alpha, A+\alpha]_{LI}x = [y-\eta, y+\eta]\ . \tag{4}$$

An $\overline{x} \in X$ satisfying all equations of (4) does not exist in general. It suffices if $\overline{x} \in X$ satisfies at least one equation of (4). So we define

$\overline{x} \in X$ is an acceptable solution of problem (4) iff

$$\exists\ \overline{A} \in [A-\alpha, A+\alpha]_{LI},\quad \overline{y} \in [y-\eta, y+\eta]:\ \overline{A}\overline{x} = \overline{y}\ . \tag{5}$$

Notice that the formulas (3) and (5) are identical. For the special case of equations in L^p - spaces we shall present a means to check whether or not an $\bar{x} \in X$ is acceptable.

II. Lemma

In this section we start anew. The notation here does not depend on section I.

Let X be a Riesz - space and let Y be a Dedekind - complete Riesz - space. Further let $L(X,Y)$ denote the set of all linear operators $X \to Y$, ordered by

$$S \leq T \quad :\Longleftrightarrow \quad Sx \leq Tx \quad \text{for all } x \in X^+ . \tag{6}$$

The subset $LB \subseteq L(X,Y)$ of all linear order - bounded operators $X \to Y$ is a Dedekind - complete Riesz - space. For the definitions and properties of Riesz - spaces we refer to Luxemburg/Zaanen [2] and Vulikh [6]. The following lemma is a generalization of the Oettli/Prager - Theorem [3].

Lemma

For $\bar{x} \in X$, $y \in Y$, $\eta \in Y^+$, $A \in LB$, $\alpha \in LB^+$ the following assertions are equivalent

(a) $\exists\ \bar{A} \in [A-\alpha, A+\alpha]_{LB},\ \ \bar{y} \in [y-\eta, y+\eta]:\ \bar{A}\bar{x} = \bar{y}$

(b) $|A\bar{x} - y| \leq \alpha|\bar{x}| + \eta$

Proof.

(a) ⇒ (b): This implication is simple to prove.

From (a) we know $|\bar{A} - A| \leq \alpha$ and $|\bar{y} - y| \leq \eta$. So we get

$|A\bar{x} - y| = |A\bar{x} - \bar{A}\bar{x} + \bar{y} - y| \leq |(A-\bar{A})\bar{x}| + |\bar{y} - y| \leq |A - \bar{A}|\cdot|\bar{x}| + \eta \leq \alpha|\bar{x}| + \eta$

(b) ⇒ (a): The Oettli/Prager proof can not be used.

The case $\bar{x} = 0$ is trivial: put $\bar{A} := A$ and $\bar{y} := 0$. In the following we assume $\bar{x} \neq 0$.

(b) supplies an interval for $A\bar{x}$,

$$A\bar{x} \in [y - \alpha|\bar{x}| - \eta, y + \alpha|\bar{x}| + \eta] = [-\alpha|\bar{x}|, \alpha|\bar{x}|] + [y-\eta, y+\eta] . \tag{7}$$

Here we use a well - known formula for adding intervals in a Riesz - space, see Schaefer [4] p. 207. From (7) we conclude

$$\exists\ \bar{z} \in [-\alpha|\bar{x}|,\alpha|\bar{x}|],\quad \bar{y} \in [y-\eta,y+\eta]:\ A\bar{x} = \bar{z} + \bar{y}\ . \tag{8}$$

Setting B: $\mathbb{R}\bar{x} \to Y$ linear with $B\bar{x} := -\bar{z}$, we obtain

$$A\bar{x} + B\bar{x} = \bar{y}\ . \tag{9}$$

The operator B is only defined on the one-dimensional linear subspace $\mathbb{R}\bar{x}$ of X.

Now we need a suitable extension of B.

The mapping s: $X \to Y$ with $s(x) := \alpha|x|$ is sublinear and for $x = \xi\bar{x} \in \mathbb{R}\bar{x}$ we get

$$Bx \le |Bx| = |B(\xi\bar{x})| = |\xi|\cdot|B\bar{x}| = |\xi|\cdot|\bar{z}| \le |\xi|\cdot\alpha|\bar{x}| = \alpha|\xi\bar{x}| = \alpha|x| = s(x).$$

Here we use $B\bar{x} = -\bar{z}$ (according to the definition of B) and $|z| \le \alpha|\bar{x}|$, which follows from (8). Thus we have

B: $\mathbb{R}\bar{x} \to Y$ linear with $Bx \le s(x)$ for all $x \in \mathbb{R}\bar{x}$.

Using the Hahn/Banach extension theorem for linear operators into Dedekind-complete Riesz-spaces, see Jameson [1] p. 64, we extend B to

$\bar{B}$: $X \to Y$ linear with $\bar{B}x \le s(x)$ for all $x \in X$.

We deduce some properties of the operator $\bar{B}$.

$\bar{B}x \le s(x) = \alpha|x| = \alpha x$ for all $x \in X^+$, so $\bar{B} \le \alpha$.

$(-\bar{B})x = \bar{B}(-x) \le s(-x) = \alpha|-x| = \alpha|x| = \alpha x$ for all $x \in X^+$, so $-B \le \alpha$.

Hence $|\bar{B}| = \sup\{\bar{B},-\bar{B}\} \le \alpha$ and $\bar{B} \in LB$.

Now we rewrite (9) using the extension $\bar{B}$ of B,

$$A\bar{x} + \overline{Bx} = \bar{y}$$

$$(A+\bar{B})\bar{x} = \bar{y}$$

If we set $\bar{A} := A + \bar{B}$, we obtain

$$\overline{Ax} = \bar{y}$$

and in addition $|\bar{A} - A| = |\bar{B}| \le \alpha$, $\bar{A} \in [A-\alpha,A+\alpha]$. QED

III. Special Case

Let M denote the Riesz - space of all real Lebesgue - measurable functions on $[0,1]$ with the usual identification of functions equal a.e. The subspaces $L^p \subseteq M$, $1 \le p < \infty$, of p-th power Lebesgue - integrable functions are Dedekind - complete Riesz - spaces and order - ideals in M. In the sequel let $1 \le p,q < \infty$. The set $L(L^p,L^q)$ of all linear operators $L^p \to L^q$ is ordered as defined in section II.

As far as linear <u>integral</u> operators $L^p \to L^q$ are concerned we follow Schep [5].

<u>Definition.</u> The linear operator $U\colon L^p \to L^q$ is called <u>integral</u> if there exists a Lebesgue - measurable function k on $[0,1]^2$ such that

a) $(Ux)(s) = \int_0^1 k(s,t)x(t)\,dt$ a.e. on $[0,1]$ for all $x \in L^p$,

b) $\int_0^1 |k(s,t)x(t)|\,dt$ represents an element of L^q for all $x \in L^p$.

The subset $LI \subseteq L(L^p,L^q)$ of all linear integral operators $L^p \to L^q$ carries the order inherited from $L(L^p,L^q)$.

Now we consider the linear interval integral equation

$$[A-\alpha, A+\alpha]_{LI}\, x = [y-\eta, y+\eta]_{L^q}\,, \tag{10}$$

where $A \in LI$, $\alpha \in LI^+$, $y \in L^q$, $\eta \in L^{q+}$ are given.

In the following theorem we show that an $\overline{x} \in L^p$ is an acceptable solution of the problem (10) iff $|A\overline{x} - y| \le \alpha|\overline{x}| + \eta$.

<u>Theorem.</u>

For $\overline{x} \in L^p$, $y \in L^q$, $\eta \in L^{q+}$, $A \in LI$, $\alpha \in LI^+$

the following assertions are equivalent

(a) $\exists\, \overline{A} \in [A-\alpha, A+\alpha]_{LI},\ \overline{y} \in [y-\eta, y+\eta]_{L^q}\colon \overline{A}\overline{x} = \overline{y}$

(b) $|A\overline{x} - y| \le \alpha|\overline{x}| + \eta$.

Proof. We use our Lemma and a result of Schep [5]. Schep showed that LI is a band in the Riesz-space LB of all linear order-bounded operators $L^p \to L^q$.

(a) ⇒ (b): Because of LI $\subseteq$ LB the operator $\bar{A}$ is order-bounded. Then proceed as in (a) ⇒ (b) of the Lemma.

(b) ⇒ (a): Because of LI $\subseteq$ LB, we may apply the Lemma. This yields

$$\exists\ \bar{A} \in [A-\alpha, A+\alpha]_{LB},\quad \bar{y} \in [y-\eta, y+\eta]_{L^q}:\ \bar{A}\bar{x} = \bar{y} \tag{11}$$

It is clear that $[A-\alpha, A+\alpha]_{LI} \subseteq [A-\alpha, A+\alpha]_{LB}$.

This inclusion is in fact an equality:

An arbitrary element C in $[A-\alpha, A+\alpha]_{LB}$ can be written in the form $C = A + B$ with $B \in LB$, $|B| \le \alpha$. LI as a band in LB is in particular an order-ideal in LB. So $|B| \le \alpha$ implies $B \in LI$. From $C = A + B \in LI$ and $|B| \le \alpha$ we get $C \in [A-\alpha, A+\alpha]_{LI}$.

Hence $[A-\alpha, A+\alpha]_{LI} = [A-\alpha, A+\alpha]_{LB}$.

With this information at hand we rewrite (11):

$$\exists\ \bar{A} \in [A-\alpha, A+\alpha]_{LI},\quad \bar{y} \in [y-\eta, y+\eta]_{L^q}:\ \bar{A}\bar{x} = \bar{y}\ .$$

QED

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