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A New Technique in Systems Analysis under Interval Uncertainty and Ambiguity

To Prof. Dr. Miguel Á. Sainz on occasion of his 60th birthday

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Abstract. The main subject of this work is mathematical and computational aspects of modeling of static systems under interval uncertainty and/or ambiguity. A cornerstone of the new approach we are advancing in the present paper is, first, the rigorous and consistent use of the logical quantifiers to characterize and distinguish different kinds of interval uncertainty that occur in the course of modeling, and, second, the systematic use of Kaucher complete interval arithmetic for the solution of problems that are minimax by their nature. As a formalization of the mathematical problem statement, concepts of *generalized solution sets* and *AE-solution sets* to an interval system of equations, inequalities, etc., are introduced. The major practical result of our paper is the development of a number of techniques for inner and outer estimation of the so-called AE-solution sets to interval systems of equations. We work out, among others, *formal approach*, *generalized interval Gauss-Seidel iteration*, *generalized preconditioning* and *PPS-methods*. Along with the general nonlinear case, the linear systems are treated more thoroughly.

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1. Introduction

What is Interval Analysis? Every scientific discipline is known to be characterized by its separate *subject matter* and specific *method*. To our mind, Interval Analysis is a joint child of mathematics and computer science

- that deals with the problems involving, on input/output or somewhere at the intermediate stages, interval or, more generally, bounded and set-membership uncertainties,
- whose distinctive feature is treating uncertainty sets as entire objects through establishing arithmetical and analytical operations, relations, etc., between them.

Interval Analysis and its specific methods are thus of highest value for the problems in which the uncertainty and ambiguity arise from the very beginning, being an inalienable part of the problem statement. In particular, Interval Analysis is in no way reduced to the so-called validated numerics, self-validating computations, etc., where the interval methods are merely an auxiliary tool for the solution of problems that are noninterval by their nature.

It was the clear consciousness of these facts that guided the author in writing down the present work, which is devoted, according to the formal title, to mathematical and computational aspects of systems modeling under uncertainty and ambiguity represented in the interval form, but, in point of fact, the contents of the paper is not exhausted by applied considerations. We avail ourselves of the practical problem statement mainly as grounds for a wider discussion, refinement

of the notions of an *interval problem*, of a *solution to an interval problem* and of other fundamental concepts, as well as a starting point for the development of a number of powerful computational techniques for inner and outer estimation of the solution sets to interval equations. We make an attempt to consider interval static systems with both general nonlinear input-state-output relationship and a simpler linear case. To some extent, this paper is a survey of earlier works by the author and other researchers, but a considerable part of the results presented is new and has not been published anywhere.

In our work, the main mathematical object under study is an *interval system of equations* of the form

$$\begin{cases} f_1(\mathbf{a}_1, \ldots, \mathbf{a}_l, x_1, \ldots, x_n) = \mathbf{b}_1, \\ f_2(\mathbf{a}_1, \ldots, \mathbf{a}_l, x_1, \ldots, x_n) = \mathbf{b}_2, \\ \qquad\qquad\qquad\qquad\quad \vdots \\ f_m(\mathbf{a}_1, \ldots, \mathbf{a}_l, x_1, \ldots, x_n) = \mathbf{b}_m, \end{cases} \tag{1.1}$$

with intervals $\mathbf{a}_1, \ldots, \mathbf{a}_l, \mathbf{b}_1, \ldots, \mathbf{b}_m$, which we also write out in a concise from

$$F(\mathbf{a}, x) = \mathbf{b} \tag{1.2}$$

with

$$F = \begin{pmatrix} F_1(\mathbf{a}, x) \\ F_2(\mathbf{a}, x) \\ \vdots \\ F_m(\mathbf{a}, x) \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and interval vectors

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_l \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{pmatrix}.$$

The interval systems of equations (1.1) are understood just as *records denoting the families of the point systems of equations of the same structure constituted by independent varying of the parameters* $a_1, \ldots, a_l, b_1, \ldots, b_m$ *within the corresponding intervals* $\mathbf{a}_1, \ldots, \mathbf{a}_l, \mathbf{b}_1, \ldots, \mathbf{b}_m$.

The major results presented in the paper relates not to the general nonlinear systems of the form (1.1)–(1.2), but to a simpler (although not less significant) *interval linear systems*

$$\begin{cases} \mathbf{a}_{11}x_1 + \mathbf{a}_{12}x_2 + \cdots + \mathbf{a}_{1n}x_n = \mathbf{b}_1, \\ \mathbf{a}_{21}x_1 + \mathbf{a}_{22}x_2 + \cdots + \mathbf{a}_{2n}x_n = \mathbf{b}_2, \\ \quad\vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ \mathbf{a}_{n1}x_1 + \mathbf{a}_{n2}x_2 + \cdots + \mathbf{a}_{mn}x_n = \mathbf{b}_m, \end{cases} \tag{1.3}$$

with the intervals $\mathbf{a}_{ij}$ and $\mathbf{b}_i$, or briefly

$$\mathbf{A}x = \mathbf{b} \tag{1.4}$$

with an interval matrix $\mathbf{A} = (\mathbf{a}_{ij})$ and interval right-hand side vector $\mathbf{b} = (\mathbf{b}_i)$.

The present work is devoted to the solution of various problem statements for the interval systems of equations (1.1)–(1.2) and (1.3)–(1.4). However, the mathematical results proper are prefaced by examination of the process of formulating and posing the interval problems. The necessity to treat this issue at length, being quite urgent indeed, is due to its poor development in modern interval analysis as well as the general entanglement both in methodology and terminology.

The viewpoint we advocate is that, in most cases, speaking of the *solution of an interval equation* (system of equations, inequalities, etc.) on its own is incorrect. The right usage of words is to talk about the solution of this or that *problem statement* relating to the interval equation (system of equations, inequalities, etc.). In its turn, the formulation of an interval problem statement means specifying at least a *solution set* and *a way of its estimation*.

In this respect, the situation in interval analysis very much resembles that e.g. in the theory of differential equations, where one usually avoids speaking of the solution of a differential equation on its own, in general. Instead, people consider, investigate and solve specific problem statements, such as "initial-value problem," "boundary-value problem" (for ordinary differential equations), "initial-value problem," "Dirichlet boundary-value problem," "Neumann boundary-value problem," "mixed problems," "radiation problem," and so on (for partial differential equations).

Our notation follows mainly the internationally adopted recommendations that has been summarized in [45].* In particular, we denote intervals and interval objects (vectors, matrices) by boldface letters (for instance, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$), while underscores and overscores—$\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$—designate the lower and upper endpoints of the interval $\mathbf{x}$.

2. Generalized Solution Sets

2.1. Description of a Practical Situation

Our main practical example will be the so-called *inverse problem*** of the systems analysis for a static (inertialess) input-state-output type system:

> *Given the input and output of a system, find (or somehow estimate) its states.*

* See also `http://www.cs.utep.edu/interval-comp/notations/suggestion.html` or `http://www.mat.univie.ac.at/~neum/software/int`.

** Often referred to as *identification problem*.

The peculiarity of the situation we will deal with is that the input and output of the system are not specified exactly, they are only supposed to be within some bounds, lower and upper, or, which is equivalent, we are given merely intervals of their possible values (variations).

Let the system state, input signal and output response be described by real vectors $x \in \mathbb{R}^n$, $a \in \mathbb{R}^l$, and $b \in \mathbb{R}^m$ respectively. In the set of all inputs, we distinguish between

- *perturbations* $a_1, \ldots, a_r$, which act within intervals $\mathbf{a}_1, \ldots, \mathbf{a}_r$ independently of our will, and
- *controls* $a_{r+1}, \ldots, a_l$, which we ourselves can choose from intervals $\mathbf{a}_{r+1}, \ldots, \mathbf{a}_l$.

The perturbations disturb the system, while appropriate controls are to compensate them and to facilitate reaching the required functioning characteristics. In classical control theory, all the system outputs that ought to be maintained at predetermined value or varied according to a predetermined plan are known to be called *regulated outputs*. However, involving intervals to describe the end use of the system outputs introduces a specific character into the situation under study. Namely, we should divide the set of all the system outputs into

- the components $b_1, b_2, \ldots, b_s$ that we must be able to transform to any values from prescribed attainability intervals $\mathbf{b}_1, \ldots, \mathbf{b}_s$, and
- the components $b_{s+1}, \ldots, b_m$ that must certainly fall into some intervals $\mathbf{b}_{s+1}, \ldots, \mathbf{b}_m$.

The outputs of the first type may be termed as *controlled* while the outputs of the second type will be called *stabilized*.

The examples of the controlled outputs are coordinates of a robotic arm or of a manipulator, which are required to "cover" with guarantee each point of a given operating area. As this covering takes place, we usually do not mind if the arm (manipulator) could additionally attain some other (extra) positions outside the operating area.

The typical example of the stabilized system output is the temperature inside a chemical reactor in a number of technological processes. It must not differ from a nominal one, T, greater than some prescribed magnitude δT, but every temperature from the interval $[T - \delta T, T + \delta T]$ is equally acceptable and the specific value of the actual temperature t does not matter provided that the membership $t \in [T - \delta T, T + \delta T]$ holds true. In particular, some of the values from $[T - \delta T, T + \delta T]$ may turn out unattainable by the process in reality.

In the system under study, the input-state-output relationship is assumed to be of the form

$$F(a, x) = b \tag{2.1}$$

with a map $F : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^m$. In general, F may have quite a sophisticated form, but in the major part of our paper we will regard the components $F_i(a, x)$,

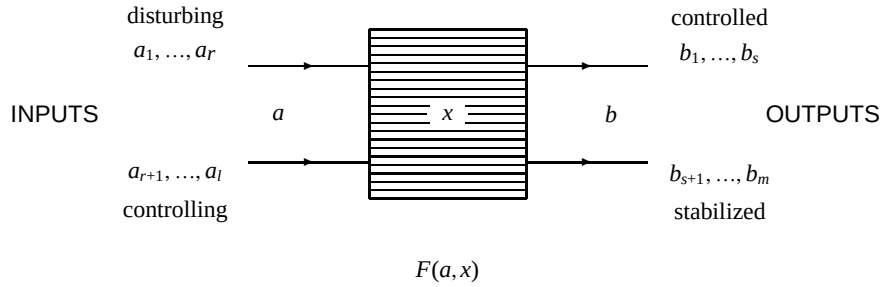

Figure 1. A structural scheme of a static control system.

$i = 1, 2, \ldots, m$, as *rational expressions*, that is, as finite combinations of the variables a, x and constants with elementary arithmetical operations (cf. [68], [69]). Also, we suppose all F_i to be continuous over their domains, that is, division by zero does not occur in $F_i(a, x)$ within the intervals $\mathbf{a}_1, \ldots, \mathbf{a}_l$ and the range of x considered. Overall, the situation is described by the structural scheme presented in Figure 1.

It is worth noting that the above situation in which we make use of the terms *control*, *regulation*, *controlling*, etc., does not entirely coincide with that in which these notions are used in classical control theory. It is customary that the control theory is applied to dynamical systems, either time-continuous or time-discrete. However, the development of the general systems theory has lead to the understanding that the dependence on the time variable is of secondary importance in the definitions of "control" and "controllability" (see, e.g., [66]). That is especially pronounced in the abstract mathematical statements of the dynamic control problems, where phase trajectories, phase constraints, admissible control actions, etc., are elements of functional spaces. In the most general form, the notion of controllability of a system (of a parameterized map) is closely related to the notion of *attainability*.

Namely, M. Mesarovic and Ya. Takahara [66] formulate controllability as the condition that every element from a marked subset of the codomain of the map can be attained (covered) provided that we appropriately chose the parameters and arguments of the map. More precisely, let the function $\Phi(c)$ describe a final result of the system operation depending on the control c. Then the system is (completely) *controllable* if and only if the following condition is valid:

> *For every final state R from a marked set*
> *there exists a control action C from a feasible area*
> *such that* $R = \Phi(C)$.

But in such a form the controllability notion is equally applicable to static systems as well, in which the time variable and the time interval do not figure at all (see, e.g., [108]).

Besides, automatic control theory is not the only scientific discipline that has to do with "controls." In particular, the sense in which we use the term "control" (and

related terms) is in good agreement with the terminology of operations research. Recall the following generally adopted definition [1], [67]: an *operation* is a purposeful action that can be characterized as

$$U = f(X, Y),$$

where U is *utility* or the value of a criterion that represents quality of the system functioning, X is a vector of variables that we can *control*, and Y is a vector of variables that cannot *be controlled* (i.e., they are uncontrolled, or, to put this another way, *disturbing*). Anyway, our use of words is quite legitimate.

Another comment. Strictly speaking, the word "uncertainty" that we use in connection with the controlling inputs is not quite adequate to the practical sense we mean by interval bounds of their possible variations. For example, one can hardly speak of the "uncertainty" with respect to the intervals representing the ranges of aircraft rudder and elevator. Still, we will further use the word "uncertainty" to keep a uniform terminology, bearing in mind either our ignorance (lack of information) or nonuniqueness (ambiguity) of the possible values as in the above aircraft control example.

2.2. Preliminary Problem Statement

Various kind of question can arise in connection with the system presented in the preceding subsection. In our work, we study the following mathematical problem statement—the problem of guaranteed set-membership estimation of the system state from its inputs and outputs:

> *For what system states x can we choose, for any perturbations $a_1 \in \mathbf{a}_1, \ldots, a_r \in \mathbf{a}_r$ and for any a priori given output values $b_1 \in \mathbf{b}_1, \ldots, b_s \in \mathbf{b}_s$, the corresponding input controls $a_{r+1} \in \mathbf{a}_{r+1}, \ldots, a_l \in \mathbf{a}_l$ such that the output response of the system $F(x, a)$ would be exactly equal to $b_1, \ldots, b_s$ in the controlled outputs and would be inside $\mathbf{b}_{s+1}, \ldots, \mathbf{b}_m$ in the stabilized outputs?* (2.2)

Solving set-membership identification problems in the interval (and even error-bounded) context is not a novelty nowadays. However, the main distinctive feature of the problems we deal with as compared with those exposed, for example, in [127] is that we consider various and different from each other kinds of uncertainty: controls-perturbations, etc.

If all the inputs and outputs of the system were determined precisely, the solution of the problem (2.2) would reduce to the solution of the equation (2.1) with respect to x. That is not at all the case when the input and output values have interval uncertainty, but, in conformity with the terminology tradition of interval analysis, we shall speak of the problem (2.2) that "the interval system of equations

$$F(\mathbf{a}, x) = \mathbf{b} \tag{1.1}$$

with the interval parameters $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_l)^\top \in \mathbb{IR}^l$ and $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)^\top \in \mathbb{IR}^m$ is under consideration." It is necessary to stress, however, that the interval system of equations (1.1) by itself should be understood only as a formal designation for a collection of point systems

$$F(a, x) = b$$

with the coefficients $a \in \mathbf{a}$ and $b \in \mathbf{b}$, nothing more. In particular, we do not even have the right to perform any transformations with it (rearrange, reduce similar terms and such like) unless we specify what is behind a "solution" to the equation (1.1), etc., and in what sense we ought to understand the equivalence of transformations with (1.1). Some words clarifying the problem statement are thus indispensable at the point, and we define first what shall be meant by the "solution set" to the system (1.1).

Getting started, it makes sense to reformulate the verbal statement of the main problem (2.2) in a more strict and formalized way. To do that, we shall use the language of the first order *predicate calculus* with the logical quantifiers "$\forall$" (universal quantifier, "for all") and "$\exists$ (existential quantifier, "exists") [20], [50]. In particular, the condition

> *for any* $a_1 \in \mathbf{a}_1, \ldots, a_r \in \mathbf{a}_r$ *and for any* $b_1 \in \mathbf{b}_1, \ldots, b_s \in \mathbf{b}_s$, *there exist* $a_{r+1} \in \mathbf{a}_{r+1}, \ldots, a_l \in \mathbf{a}_l$ *such that* $F_1(a, x), \ldots, F_s(a, x)$ *are equal to* $b_1, \ldots, b_s$ *and* $F_{s+1}(a, x), \ldots, F_m(a, x)$ *are inside* $\mathbf{b}_{s+1}, \ldots, \mathbf{b}_m$,

which is a cornerstone of the problem statement (2.2), has to be equivalently rewritten as the following predicate (logical formula):

$$\begin{aligned}&(\forall a_1 \in \mathbf{a}_1) \cdots (\forall a_r \in \mathbf{a}_r)\ (\forall b_1 \in \mathbf{b}_1) \cdots (\forall b_s \in \mathbf{b}_s)\\ &(\exists a_{r+1} \in \mathbf{a}_{r+1}) \cdots (\exists a_l \in \mathbf{a}_l)\ (\exists b_{s+1} \in \mathbf{b}_{s+1}) \cdots (\exists b_m \in \mathbf{b}_m)\ \ \big(F(a, x) = b\big).\end{aligned} \tag{2.3}$$

To sum up, the set of all states x satisfying the question of the problem (2.2) (we will designate it by Ξ) is described as follows

$$\begin{aligned}\Xi := \{\, x \in \mathbb{R}^n \mid\ &\\ &(\ \ \forall a_1 \in \mathbf{a}_1\ \ ) \cdots (\forall a_r \in \mathbf{a}_r)(\ \ \forall b_1 \in \mathbf{b}_1\ \ ) \cdots (\forall b_s \in \mathbf{b}_s\ )\\ &(\exists a_{r+1} \in \mathbf{a}_{r+1}) \cdots (\ \exists a_l \in \mathbf{a}_l\ )(\exists b_{s+1} \in \mathbf{b}_{s+1}) \cdots (\exists b_m \in \mathbf{b}_m)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(F(a, x) = b\big)\,\},\end{aligned} \tag{2.4}$$

while the main problem under consideration can be reformulated as

> *Find (or somehow estimate) the set* Ξ *defined by (2.4).*

Notice that the definition (2.4), the most correct mathematically, is arranged according to the *separation axiom* of the formal set theory ZFC (after *Zermelo-Fraenkel-axiom of Choice*, see, e.g., [20], [50], [54], [123]). Namely, a point $\tilde{x}$ belongs to the

set (2.4) if and only if substituting it for the variable x in the predicate (2.3) results in a true proposition. In other words, the property (2.3) which is written out, as a predicate, after the vertical line in the record (2.4) "separates" some values of x that constitute the solution set.*

DEFINITION 2.1. The logical formula written out after the vertical line in the definition of the set (2.4), which thus determines a characteristic property of the points of this set, will be called *separating predicate* of the corresponding set (2.4).

We emphasize that, apart from setting the function F and interval vectors $\mathbf{a}$ and $\mathbf{b}$, the keystone in the definition (2.4) is our indicating the quantifiers $\forall$ and $\exists$ at various parameters a and b of the system (1.1). Another point is that the set Ξ determined by (2.4) has all the rights to be referred to as a *solution set* of the interval system of equations (1.1) as, say, traditionally understood solution set formed by *all* the solutions x to the point equations $F(a, x) = b$ with $a \in \mathbf{a}$ and $b \in \mathbf{b}$ (the latter is called *united solution set*, see Section 3.1). Basically, (2.4) is a solution set in some generalized sense which we are going to discuss in the forthcoming sections. We will refer to the solution sets determined by (2.4) and similar definitions involving occurrences of different logical quantifiers as *generalized solution sets* of interval equations systems.

2.3. Quantifier Formalism

Let us summarize what has been done in the previous subsections. Taking the inverse system analysis problem (2.2) as a prototype, we have realized the necessity to consider the solution set of the form (2.4). In doing that, we applied the universal and existential quantifiers to the system inputs a_j, which are only known to belong to some intervals, to express the principal distinction between

- the inputs that are not under our will, being external uncontrolled disturbances (this corresponds to the record "$\forall a_j \in \mathbf{a}_j$"),

  and

- the inputs that we are able to vary within prescribed intervals by our will, i.e., to control (this corresponds to the record "$\exists a_j \in \mathbf{a}_j$").

With respect to the system outputs b_i, the logical quantifiers was applied to distinguish between

- stabilization corridors of the system within which it is required to ensure functioning of the system irrespective of values of the disturbances (this corresponds to the record "$\exists b_i \in \mathbf{b}_i$"),

* Some authors use the terms "selection," "segregation" instead of "separation." Besides, the construction we pointed out determines, strictly speaking, infinitely many "separation axioms" obtained by fixing this or that specific predicate, so that one may read of the "separations axioms" or even "axiom schema of separation" in the fundamental treatises on the subject.

and

- attainability sets of the system whose every element is to be covered as the result of an appropriate choice of the controlled factors (this corresponds to the record "$\forall b_i \in \mathbf{b}_i$").

However, the mathematical object described by the definition (2.4) has a separate importance on its own, and one could arrive at introducing the general definition of the solution sets (2.4) from an abstract standpoint as well, without invoking practical analysis of intervally defined systems that we dwell on in Sections 2.1 and 2.2.

As is fairly simple to realize, a dual character is inherent to the very understanding and interpretation of the interval (and, more generally, set-membership) uncertainty. The point is that in real-life problems one is hardly interested in intervals by themselves, as integral and undivided objects, with no further internal structure. In most cases, we only use an interval $\mathbf{v}$ in connection with a property (let us denote it by P) that can be fulfilled or not for its point members. Under the circumstances, the following different situations may occur:

- either the property $P(v)$ considered (that may be a point equation, inequality, etc.) holds for *all* members v from the given interval $\mathbf{v}$,
- or the property $P(v)$ holds only for *some* members v from the interval $\mathbf{v}$, not necessarily all (maybe, only for one value).

The above stated may mean, in particular, that in the first case *all* values from an interval are possible, while in the second case the interval means only bounds on the (unknown) value, i.e., that *some*, not necessarily all, values are possible, and these possible values belong to the given interval. The distinction between the two types of the interval uncertainty is especially pronounced when a system has several varying parameters that describe actions of different nature, which pursue different goals and may conflict with each other (like disturbances-controls).

In formal writing, the above distinction is manifested in using the logical quantifiers—either the universal quantifier "$\forall$" or the existential quantifier "$\exists$":

- in the first case, we write "$(\forall v \in \mathbf{v})\ P(v)$" and shall speak of $\forall$*-type (A-type) of uncertainty*,
- in the second case, we write "$(\exists v \in \mathbf{v})\ P(v)$" and are going to speak of $\exists$*-type (E-type) of uncertainty*

(see also [99], [100], [104], [105], [107], [110]–[112], [117], [124]).

It is worthwhile to stress that our reasoning justifying the use of logical quantifiers with respect to intervally uncertain parameters are equally applied not only to interval algebraic systems of the form (1.1), but also to interval inequalities, interval differential equations, integral equations and so forth. When strictly defining solutions and solution sets to all these problems, we should consciously take into account the difference between the interval uncertainty types. Specific examples are in order.

Let us consider an object described by a system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, v), \tag{2.5}$$

$$t \in [0, T], \quad x(0) = x_0, \tag{2.6}$$

where

t is time variable,
$x(t)$ is a phase state vector,
$v(t)$ is a control vector which is assumed to be within some interval $\mathbf{V} \in \mathbb{R}^p$, i.e., $v(t) \in \mathbf{V}$ for all $t \in [0, T]$.

Attainability set of the system (2.5)–(2.6) is known to be [40], [61] the set of all the endpoints $x(T)$ of the trajectories of the system issuing from the point x_0 and corresponding to various possible values of the control $v(t)$, that is, the set

$$\Big\{x(T) \mid \big(x(0) = x_0\big) \;\&\; \big(\exists\, v(t) \in \mathbf{V}\big) \;\; \Big(\dot{x} = f\big(t, x(t), v(t)\big)\Big)\Big\}.$$

A more complex situation is when the object under consideration undergoes uncontrolled perturbation (noise) $u(t) \in \mathbf{U}$, $\mathbf{U} \in \mathbb{R}^q$, so that its mathematical model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, u, v), \tag{2.7}$$

$$t \in [0, T], \quad x(0) = x_0, \tag{2.8}$$

rather than (2.5)–(2.6). It is common knowledge that controlling a dynamical object, aimed at achieving some objectives and/or optimizing a quality criterion, can be conducted in one of the two alternative ways. Namely,

- the control action may be taken from a predetermined *program* calculated on the base of an *a priori* information about the system,

  or, otherwise,

- the control action may be formed as the result of a *position-based strategy* in which the control action is calculated *a posteriori*, relying on additional information about the system we learn during the process.

In practice, the first of these alternatives corresponds to the control according to a rigid scenario, while the second one is nothing but feedback regulation. Studying and constructing position-based controls in dynamical systems is the subject matter of *differential games* theory (see [37], [67]), an exciting (although not quite simple) mathematical discipline into which we shall not delve deeply in our work. Still, within the position approach the following "global" question makes sense:

> *What is the set of the final points $x(T)$ to which, regardless of a specific noise realization $u(t) \in \mathbf{U}$, the initial position $x(0)$ can be transferred by appropriate choice of the control $v(t) \in \mathbf{V}$?*

The set of points meeting the above requirement is exactly

$$\Big\{x(T) \mid \big(x(0) = x_0\big) \ \&\ \big(\forall\, u(t) \in \mathbf{U}\big)\big(\exists\, v(t) \in \mathbf{V}\big) \Big(\dot{x} = f\big(t, x(t), u(t), v(t)\big)\Big)\Big\}, \tag{2.9}$$

that is, in fact, may be characterized as a generalized solution set, as we have defined them, to the interval system of differential equations (2.7)–(2.8).

Summing up, we see in these examples that the control design problem can be equivalently reformulated as the problem of finding points from the set (2.9) of the solution, in some generalized sense, which is constructed using our quantifier formalism. Overall, the above ideas as applied to the interval differential equations are still waiting to be elaborated and put into practice. Some tentative, but very promising examples of fruitful applications of the quantifier language to control system design can be found, e.g., in [39]. Meanwhile, an experience of study of this kind of interval inequalities and interval optimization problems does exist.

A. Vatolin [124] was the pioneer in researching interval optimization problems with quantified interval uncertainty (see also [21]). For interval linear system of inequalities

$$\mathbf{A}x \le \mathbf{b}, \tag{2.10}$$

A. Voshinin and G. Sotirov [126] seem to be the first who considered the solution sets

$$\begin{aligned}
&\{x \in \mathbb{R}^n \mid (x \ge 0) \ \&\ (\forall A \in \mathbf{A})\,(\forall b \in \mathbf{b}) \ \ (Ax \le b)\},\\
&\{x \in \mathbb{R}^n \mid (x \ge 0) \ \&\ (\exists A \in \mathbf{A})\,(\forall b \in \mathbf{b}) \ \ (Ax \le b)\},\\
&\{x \in \mathbb{R}^n \mid (x \ge 0) \ \&\ (\forall A \in \mathbf{A})\ (\exists b \in \mathbf{b}) \ \ (Ax \le b)\},\\
&\{x \in \mathbb{R}^n \mid (x \ge 0) \ \&\ (\exists A \in \mathbf{A})\ (\exists b \in \mathbf{b}) \ \ (Ax \le b)\}
\end{aligned}$$

in connection with linear constrained optimization problems under interval uncertainty. Later, J. Rohn and J. Kreslová [90] studied the notions of *weak solvability* and *strong solvability* for interval linear inequalities (2.10):

- a system (2.10) is called *weakly solvable* if for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ the point system $Ax \le b$ has a solution (which generally depends on A and b);
- a system (2.10) is called *strongly solvable* if there exists a solution $\tilde{x}$ satisfying the point system $Ax \le b$ for each $A \in \mathbf{A}$, $b \in \mathbf{b}$.

It is not hard to see that the strong solvability of the interval inequalities system $\mathbf{A}x \le \mathbf{b}$ is nothing but the property of the set

$$\{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\forall b \in \mathbf{b}) \ \ (Ax \le b)\},$$

which is one of the generalized solution sets to $\mathbf{A}x = \mathbf{b}$, to be nonempty.

Overall, the mathematical object defined by the record (2.4) has a significance of its own, and it makes sense to single it out as a separate notion. But, before

doing this, it should be recognized that the definition (2.4) is not the most general one yet. Since the different quantifiers do not commute with each other, we can form another solution sets through combining "$\forall$" and "$\exists$" with the parameters and changing their order!

For example, given a one-dimensional interval equation

$$\phi(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, x) = \mathbf{b}$$

with four interval parameters in the left-hand side function, one can consider, as a solution set,

$$\{x \in \mathbb{R}^n \mid (\exists a_2 \in \mathbf{a}_2)(\forall a_1 \in \mathbf{a}_1)(\forall a_4 \in \mathbf{a}_4)(\forall b \in \mathbf{b})(\exists a_3 \in \mathbf{a}_3) \\ \big(\varphi(a_1, a_2, a_3, a_4, x) = b\big)\},$$

or

$$\{x \in \mathbb{R}^n \mid (\forall a_1 \in \mathbf{a}_1)(\forall a_2 \in \mathbf{a}_2)(\exists a_4 \in \mathbf{a}_4)(\forall b \in \mathbf{b})(\exists a_3 \in \mathbf{a}_3) \\ \big(\varphi(a_1, a_2, a_3, a_4, x) = b\big)\},$$

and so on.

Prior to formulating the general definition, let us remind that the most profound generalization of the concept of a system of equations (inequalities, etc.) is the so-called *constraint satisfaction problem* that emerged in the researches on artificial intelligence [64] in the late 70s. We will need a somewhat updated definition of a *numeric constraint satisfaction problem*:

DEFINITION 2.2 [62]. A *numerical constraint satisfaction problem* is a triple $P = (V, D, C(x))$ defined by

(i) a set of numeric variables $V = \{x_1, \ldots, x_n\}$,

(ii) a set of domains $D = \{D_1, \ldots, D_n\}$ where D_i, a set of numeric values, is the domain associated with the variable x_1,

(iii) a set of constraints $C(x) = \{C_1(x), \ldots, C_m(x)\}$ where a constraint $C_i(x)$ is determined by any numeric relation (equation, inequality, inclusion, etc.) linking a set of variables under consideration.

A *solution* to a numeric constraint satisfaction problem $P = (V, D, C(x))$ is an instantiation of the variables of V for which both inclusion in the associated domains and all the constraint of $C(x)$ are satisfied.

All the solutions of the constraint satisfaction problem thus constitute the set

$$\{x \in D \mid C(x) \text{ is satisfied}\}.$$

We can complicate the situation assuming that the constraints $C_i(x)$ entering the Definition 2.2 depend on some parameters $p_1, p_2, \ldots, p_l$ about which we only know that they may belong to the intervals $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_l$. We have thus a constraint system

$C(\mathbf{p},x) = \{C_1(\mathbf{p},x),\ldots,C_m(\mathbf{p},x)\}$ with the interval parameters, and, as a consequence, an *interval constraint satisfaction problem*. Taking into account the dual character of interval uncertainty and our above observations, the most general definition of the set of solutions to such interval constraint satisfaction problem $(V,D,C(\mathbf{p},x))$ should have the form

$$\{x\in D \mid (Q_1 p_{\pi_1}\in \mathbf{p}_{\pi_1})(Q_2 p_{\pi_2}\in \mathbf{p}_{\pi_2})\cdots(Q_l\, p_{\pi_l}\in \mathbf{p}_{\pi_l}) \quad \big(C(p,x) \text{ is satisfied}\big)\}, \qquad (2.11)$$

where

Q_i are logical quantifiers $\forall$ or $\exists$,

$p = (p_1,p_2,\ldots,p_l)\in\mathbb{R}^l$
is the vector of parameters of the constraints system considered,

$\mathbf{p} = (\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_l)\in\mathbb{IR}^l$
is the interval vector of the possible values of these parameters,

$\pi = (\pi_1,\pi_2,\ldots,\pi_l)$
is a permutation of the numbers $1,2,\ldots,l$.

DEFINITION 2.3. The sets of the form (2.11) will be referred to as *generalized solution sets* to the interval constraints satisfaction problem $(V,D,C(\mathbf{p},x))$.

In particular, for the interval systems of equations of the form (1.1) we adopt

DEFINITION 2.4. *Generalized solution sets* to an interval equations system $F(\mathbf{a},x)=\mathbf{b}$ are the sets of the form

$$\{x\in\mathbb{R}^n \mid (Q_1 z_{\pi_1}\in \mathbf{z}_{\pi_1})(Q_2 z_{\pi_2}\in \mathbf{z}_{\pi_2})\cdots(Q_{l+m} z_{\pi_{l+m}}\in \mathbf{z}_{\pi_{l+m}}) \ \big(F(a,x)=b\big)\},$$

where

$Q_1,Q_2,\ldots,Q_{l+m}$
are the logical quantifiers $\forall$ or $\exists$,

$(z_1,z_2,\ldots,z_{l+m}) := (a_{11},\ldots,a_l,b_1,\ldots,b_m)\in\mathbb{R}^{l+m}$
is the aggregated (compound) parameter vector of the system of equations considered,

$(\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_{l+m}) := (\mathbf{a}_1,\ldots,\mathbf{a}_l,\mathbf{b}_1,\ldots,\mathbf{b}_m)\in\mathbb{IR}^{l+m}$
is the aggregated vector of the intervals of the possible values of these parameters,

$(\pi_1,\pi_2,\ldots,\pi_{l+m})$
is a permutation of the integers $1,2,\ldots,l+m$.

The Definitions 2.2 and 2.4 are, indeed, very general. The quantifier that corresponds to each interval element may have two values $\{\forall,\exists\}$ and the order of

the quantified terms in the separating predicate is also essential for the definition. Therefore, the total number of the solution sets we can thus define for the interval equations system (1.1) far exceeds 2^{l+m}, that is, 2 to the power of the "number of interval parameters of the system." In the general case, these solution sets can be practically interpreted as solutions of some *games* or *multistep decision-making processes* under interval uncertainty (which was first pointed out in [124]) as well as solutions to some minimax operations research problems [99], [100].

2.4. INTERPRETATION

Game theory is widely recognized to be a method of exploration of the logic of interaction between two or more rational actors. As a mathematical discipline, game theory deals with both construction and investigation of models of conflict phenomena, i.e., such ones that involve participants (called *players*) pursuing different goals by use of some *strategies*.

In game theory and in multistage decision making, we must not only describe which parameters are controllable, but also who controls specific parameters, and in what order. These control actions are usually called *moves* and to specify the sequence of moves it is natural to apply the language of the graph theory. Formally, a finite *perfect information game* (in the so-called *extensive form*) is known to be (see, e.g., [67], [78]) a quadruple

$$\langle K, X, \mathcal{R}, \{h_k\}_{k\in K}\rangle,$$

where

- K is a finite *set of players*;
- X is a finite *tree of the game* (i.e, a graph that is a tree) of which the nodes are called *positions* and the root is called *initial position*; for the positions, a succession relation is defined, so that the positions that follow a given position $x \in X$ are termed *alternatives* of x, while the positions that do not have alternatives are *final positions*, and the paths that lead to them are called *parties*; the set of final positions is usually designated by X^*;
- $\mathcal{R}$ is a partition of the set $X \setminus X^*$ to *n priority subsets* $X_1, X_2, \ldots$, whose number is equal to the cardinality of the set K, such that the k-th player makes a move in a position from X_k;
- h_k are *payoff functions* of the game, i.e., such functions that assign a payment of the k-th player to each final position.

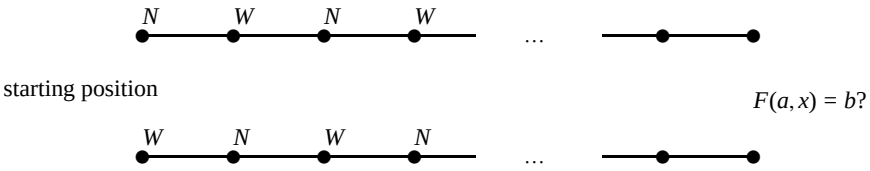

Figure 2. Game trees interpreting the generalized solution sets.

The above definition is the most general and we really do not need all its features. To interpret the generalized solution sets, it will suffice to restrict ourselves to the simplest game in which

- there are only two players,
- the tree of the game is a *simple path* (see [91]),
- the payoff functions are Boolean-valued, and
- the interests of the players—the values of their payoff functions— are diametrically opposite (such games are termed *antagonistic*).

We can therefore think of the possible outcomes $\{0, 1\}$ of the game as "loss-win," the loss of the first player being the win of the second one and vise versa.

Given a generalized solution set to the interval system of equations $F(\mathbf{a}, x) = \mathbf{x}$, let us consider such a game between the players N (*Nature*) and W (*We*), in which N has at his disposal all the interval parameters having A-uncertainty, while W controls the parameters with E-uncertainty. The players make their moves in turn, one after the other, so that the game tree is really a simple path and looks as one of those described at Figure 2 depending on which player makes the first move in the starting position. The latter is prescribed, as is easily seen, by which quantifier prefix, "$\forall$" or "$\exists$", comes first in the separating predicate of the solution set under consideration. The result of the game is determined by whether the equality $F(a, x) = b$ is finally attained or not: if the player W manage to get it, then he has won the game; otherwise, when we do not have the equality $F(a, x) = b$ in the final position, the player W has lost and the winner is N. Then, for example, the solution set

$$\{x \in \mathbb{R}^n \mid (\exists a_2 \in \mathbf{a}_2)(\forall a_1 \in \mathbf{a}_1)(\forall a_3 \in \mathbf{a}_3)(\exists a_4 \in \mathbf{a}_4)(\forall b_2 \in \mathbf{b}_2) \cdots \big(F(a, x) = b\big)\}$$

can be interpreted in the following manner: there exists such a starting move of the player W (who begins the game) by a_2 that, no matter how the player N moves, W can find an appropriate reply again, etc., so that the equality $F(a, x) = b$ will be eventually achieved.

A specific value of the variable x may be thus considered as a parameter of the game, while the generalized solution set is nothing but the set of all such x's that

We defeat the Nature in the game defined by the interval system of equations (1.1) and a specified distribution of the uncertainty types over its interval parameters.

To conclude, it is worth mentioning that in [51] V. Kreinovich et al. give a short critical overview of the quantifier formalism we are developing, point out some its successes and limitations. Meanwhile, some ideas concerning the distinction between the interval uncertainty of the A-type and E-type penetrate even the commercial interval Fortran 95 compiler recently released by Sun Microsystems Inc [23]. The careful demarcation of the so-called "certain relations," "possible relations," and "set relations" in [23] is nothing but an attempt to classify the interval binary relations—"$>$", "$<$", "$\leq$", and "$\geq$"—according to the uncertainty types the intervals under comparison bear. Namely, the "certain relations" correspond to both compared intervals being A-uncertain, the "possible relations" correspond to the E-uncertainty of the intervals, while the "set relations" matches the case when the uncertainty is mixed, i.e. one of the intervals has A-uncertainty and the other has E-uncertainty.

3. AE-Solution Sets

3.1. Definitions

In order not to be lost in a great variety of generalized solution sets, it makes sense to somewhat confine and structurize our considerations. In this work, we are going to treat (for the time being) mainly the solution sets of the form (2.4), or, in other words, the generalized solution sets with the separating predicate in which *all the occurrences of the universal quantifier* $\forall$ *precede the occurrences of the existential quantifier* $\exists$. Using the terminology of the mathematical logic, we can rephrase this condition by saying that the corresponding separating predicate must have an *AE-form*.

DEFINITION 3.1. *AE-solution sets* are generalized solution sets to interval equations (inequalities, etc.) for which the separating predicate has AE-form.

Let us consider, for the AE-solution sets, various possible ways of describing the uncertainty types distribution with respect to the interval parameters of the system:

1. As far as the order of the quantifiers is fixed, the simplest of such ways is to directly point out which quantifier is applied to this or that element of the interval system. Namely, let us introduce an l-vector $\alpha = (\alpha_i)$ and an m-vector $\beta = (\beta_i)$ made up of the logical quantifiers and such that

$$\alpha_i := \begin{cases} \forall, & \text{if } \mathbf{a}_i \text{ has A-uncertainty,} \\ \exists, & \text{if } \mathbf{a}_i \text{ has E-uncertainty,} \end{cases} \tag{3.1}$$

$$\beta_i := \begin{cases} \forall, & \text{if } \mathbf{b}_i \text{ has A-uncertainty,} \\ \exists, & \text{if } \mathbf{b}_i \text{ has E-uncertainty.} \end{cases} \tag{3.2}$$

Specifying α and β, along with the interval system itself, completely determines the corresponding AE-solution set.

2. Another way to represent the uncertainty types corresponding to the elements of the interval system of equations (1.1) is to trace out partitions of the index sets of the components of both the vectors **a** and **b**. More precisely, let the entire set of the indices i of the entries a_i, i.e., the set $\{1, 2, \ldots, l\}$ be divided into two nonintersecting parts $\hat{\Gamma} := \{\hat{\gamma}_1, \ldots, \hat{\gamma}_p\}$ and $\check{\Gamma} := \{\check{\gamma}_1, \ldots, \check{\gamma}_q\}$, $p + q = l$, such that

$$\begin{aligned} &a_i \text{ is of the interval A-uncertainty for } i \in \hat{\Gamma}, \\ &a_i \text{ is of the interval E-uncertainty for } i \in \check{\Gamma}. \end{aligned} \tag{3.3}$$

Similarly, we introduce nonintersecting sets of the integer indices $\hat{\Delta} := \{\hat{\delta}_1, \ldots, \hat{\delta}_s\}$ and $\check{\Delta} := \{\check{\delta}_1, \ldots, \check{\delta}_t\}$, $\hat{\Delta} \cup \check{\Delta} = \{1, 2, \ldots, m\}$, such that, in the right-hand side vector,

$$\begin{aligned} &b_i \text{ is of the interval A-uncertainty for } i \in \hat{\Delta}, \\ &b_i \text{ is of the interval E-uncertainty for } i \in \check{\Delta}. \end{aligned} \tag{3.4}$$

We allow the natural possibility for some of the sets $\hat{\Gamma}$, $\check{\Gamma}$, $\hat{\Delta}$, $\check{\Delta}$ to be empty. It is evident that

$$\alpha_i = \begin{cases} \forall, & \text{if } i \in \hat{\Gamma}, \\ \exists, & \text{if } i \in \check{\Gamma}, \end{cases} \qquad \beta_i = \begin{cases} \forall, & \text{if } i \in \hat{\Delta}, \\ \exists, & \text{if } i \in \check{\Delta}, \end{cases}$$

and, again, determining $\hat{\Gamma}$ and $\check{\Gamma}$, $\hat{\Delta}$ and $\check{\Delta}$ results in a complete specification of an AE-solution set to the interval system of equations (1.1).

3. The third way to describe the uncertainty types distribution for an interval system of equations is to fix *disjoint decompositions* of both the interval vectors **a** and **b**. Namely, we define interval vectors $\mathbf{a}^{\forall} = (\mathbf{a}_i^{\forall})$ and $\mathbf{a}^{\exists} = (\mathbf{a}_i^{\exists})$ and interval vectors $\mathbf{b}^{\forall} = (\mathbf{b}_i^{\forall})$ and $\mathbf{b}^{\exists} = (\mathbf{b}_i^{\exists})$, of the same sizes as **a** and **b**, as follows:

$$\mathbf{a}_i^{\forall} := \begin{cases} \mathbf{a}_i, & \text{if } \alpha_i = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{a}_i^{\exists} := \begin{cases} \mathbf{a}_i, & \text{if } \alpha_i = \exists, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

$$\mathbf{b}_i^{\forall} := \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{b}_i^{\exists} := \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \exists, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

Therefore,

$$\begin{aligned} &\mathbf{a} = \mathbf{a}^{\forall} + \mathbf{a}^{\exists}, \qquad && \mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}, \\ &\mathbf{a}_i^{\forall} \cdot \mathbf{a}_i^{\exists} = 0, && \mathbf{b}_i^{\forall} \cdot \mathbf{b}_i^{\exists} = 0 \end{aligned}$$

for all i. The vectors $\mathbf{a}^{\forall}$ and $\mathbf{b}^{\forall}$ concentrate all the interval elements of the system that correspond to the A-uncertainty, while the vectors $\mathbf{a}^{\exists}$ and $\mathbf{b}^{\exists}$ store all the elements that correspond to the interval E-uncertainty.

It should be stressed that the three groups of the objects considered, which arise in connection with an AE-solution set of an interval system (1.1), namely

1) the quantifier vectors α and β,
2) decompositions of the index sets of the vectors $\mathbf{a}$ and $\mathbf{b}$ to the nonintersecting subsets $\hat{\Gamma}$ and $\check{\Gamma}$, $\hat{\Delta}$ and $\check{\Delta}$,
3) disjoint decompositions of the interval vectors $\mathbf{a} = \mathbf{a}^{\forall} + \mathbf{a}^{\exists}$ and $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$,

are in a one-to-one correspondence, so that pointing out any one item of the above triple immediately determines the other two. We will extensively use all three descriptions and change any one for another without special explanations.

Summarizing, we can give the following

DEFINITION 3.2. Let, for the interval equation $F(\mathbf{a}, x) = \mathbf{b}$, a distribution of various uncertainty types over its interval elements be represented by the quantifier vectors α and β defined by (3.1)–(3.2), or, which is equivalent, by disjoint decompositions of the index sets of $\mathbf{a}$ and $\mathbf{b}$ defined by (3.3)–(3.4). We will call the set

$$\begin{aligned}\{x \in \mathbb{R}^n \mid \\ (\forall a_{\hat{\gamma}_1} \in \mathbf{a}_{\hat{\gamma}_1}) \cdots (\forall a_{\hat{\gamma}_p} \in \mathbf{a}_{\hat{\gamma}_p})(\forall b_{\hat{\delta}_1} \in \mathbf{b}_{\hat{\delta}_1}) \cdots (\forall b_{\hat{\delta}_s} \in \mathbf{b}_{\hat{\delta}_s}) \\ (\exists a_{\check{\gamma}_1} \in \mathbf{a}_{\check{\gamma}_1}) \cdots (\exists a_{\check{\gamma}_q} \in \mathbf{a}_{\check{\gamma}_q})(\exists b_{\check{\delta}_1} \in \mathbf{b}_{\check{\delta}_1}) \cdots (\exists b_{\check{\delta}_t} \in \mathbf{b}_{\check{\delta}_t}) \\ \big(F(a, x) = b\big)\}\end{aligned} \tag{3.7}$$

AE-solution set of the type $\alpha\beta$ to the interval system of equations $F(\mathbf{a}, x) = \mathbf{b}$ (or *set of AE-solutions of the type $\alpha\beta$*) and denote it by $\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$.[⋆]

The particular cases of the above definition are the following three solution sets which have been the subject of (more or less) active research in modern interval analysis:

- *United solution set*

$$\Xi_{uni}(F, \mathbf{a}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists a \in \mathbf{a})(\exists\, b \in \mathbf{b}) \ \big(F(a, x) = b\big)\}, \tag{3.8}$$

  formed by the solutions of all point equations systems $F(a, x) = b$ with $a \in \mathbf{a}$ and $b \in \mathbf{b}$. It is undoubtedly the most popular of the solution sets, which is no wonder due to historical origination of interval analysis from sensitivity problems. Ξ_{uni} is sometimes called simply *solution set*. Its analogue for dynamical systems is the well-known *attainability set* (see [40], [61]).

- *Tolerable solution set*

$$\Xi_{tol}(F, \mathbf{a}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall a \in \mathbf{a})(\exists\, b \in \mathbf{b}) \ \big(F(a, x) = b\big)\}, \tag{3.9}$$

  formed by all point vectors x such that the image $F(a, x) \in \mathbf{b}$ for any $a \in \mathbf{a}$ (see, e.g., [18], [48], [69], [71], [74], [119], [120]). It was actually the first of the solution sets the definition of which involves different logical quantifiers.

[⋆] In his early papers [99], [100], [104], [105], [107], [113], the author called them *$\alpha\beta$-solution sets*.

- *Controllable solution set*

$$\Xi_{ctr}(F, \mathbf{a}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists\, a \in \mathbf{a}) \ \big(F(a, x) = b\big)\}, \tag{3.10}$$

  formed by all point vectors $x \in \mathbb{R}^n$, such that for any desired $b \in \mathbf{b}$ we can find an appropriate $a \in \mathbf{a}$ satisfying $F(a, x) = b$ (see [108]).

We could have already seen a practical example of an AE-solution set in Sections 2.1 and 2.2 examining the inverse systems analysis problem. Another specific example when a set of AE-solutions to an interval equations system naturally comes into existence, let us consider the *quality control model*. Similar one has been recently studied in the interval (bounded errors) context by S. Hadjihassan, E. Walter and L. Pronzato in [30], but unlike their off-line model that examines the quality control problem only for the design stage, we turn to a more complete and realistic model that takes into account uncertainty (ignorance) both at the design and manufacturing stages.

Developing Taguchi's ideas (see, e.g., [30]), it is natural to divide the set of all factors that affect the output performance characteristics of production into three subsets:

- *design factors* $x \in \mathbb{R}^n$ whose values are to be chosen at the design stage,
- *noise factors* $u \in \mathbb{R}^p$ whose values we can neither predict at the design stage nor control during the manufacturing process, and
- *manufacturing control factors* $v \in \mathbb{R}^q$ that we are able and have to use to compensate the influence of the noise factors at the manufacturing stage to ensure the desired output characteristics.

A typical quality control problem is to attain certain target values y_i^* of given performance characteristics y_i, $i = 1, 2, \ldots, m$, while the dependency of the performance characteristic y_i on the factors x, u, v is described by a mathematical model

$$y_i = F_i(x, u, v), \qquad i = 1, 2, \ldots, m,$$

with F_i being some known functions $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$.

The only available information on the fluctuation of the noise factors is assumed to be expressed in the form of intervals of their possible values: $u_i \in \mathbf{u}_i = [\underline{u}_i, \overline{u}_i]$, $i = 1, 2, \ldots, p$. Similarly, the manufacturing factors v_i may not be arbitrary. One can only take them from some intervals $\mathbf{v}_i = [\underline{v}_i, \overline{v}_i]$, $i = 1, 2, \ldots, q$. Finally, the substantial modification of the model we deal with as compared to the model considered in [30] is that, instead of the point target values y_i^* at the output of the manufacturing process, we take entire intervals $\mathbf{y}_i = [\underline{y}_i, \overline{y}_i]$, $i = 1, 2, \ldots, m$, of feasible performance characteristics, falling into them being permitted according to the process specification and/or quality criteria. In particular, if $\underline{y}_i = \overline{y}_i = y_i^*$, we arrive at the traditional model.

Under the conditions described, the main problem with the quality control is as follows:

How can one choose the design parameters x that for any noise factors $\tilde{u}_1, \ldots, \tilde{u}_p$, which are inside the limits of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ respectively, such manufacturing control factors $\tilde{v}_1 \in \mathbf{v}_1, \ldots, \tilde{v}_q \in \mathbf{v}_q$ can be selected that the corresponding output performance characteristics $F_i(x, \tilde{u}, \tilde{v})$ would still remain within the process specifications $\mathbf{y}_i$, $i = 1, 2, \ldots, m$?

It is not too hard to realize that all such designs x form the set

$$\{x \in \mathbb{R}^n \mid (\forall u_1 \in \mathbf{u}_1) \cdots (\forall u_p \in \mathbf{u}_p)(\exists v_1 \in \mathbf{v}_1) \cdots (\exists v_q \in \mathbf{v}_q) \\ \big(F_1(x,u,v) \in \mathbf{y}_1 \cdots F_m(x,u,v) \in \mathbf{y}_m\big)\},$$

or, if we put $y = (y_1, \ldots, y_m)^\top$,

$$\{x \in \mathbb{R}^n \mid (\forall u_1 \in \mathbf{u}_1) \cdots (\forall u_p \in \mathbf{u}_p)(\exists v_1 \in \mathbf{v}_1) \cdots (\exists v_q \in \mathbf{v}_q) \\ (\exists y_1 \in \mathbf{y}_1) \cdots (\exists y_m \in \mathbf{y}_m) \ \big(F(x,u,v) = y\big)\},$$

which is exactly a generalized AE-solution set, as they have been defined above, to the system of interval equations

$$\begin{cases} F_1(x, \mathbf{u}, \mathbf{v}) = \mathbf{y}_1, \\ \qquad\qquad \vdots \\ F_m(x, \mathbf{u}, \mathbf{v}) = \mathbf{y}_m. \end{cases}$$

3.2. Characterization of AE-Solution Sets

THEOREM 3.1.

$$\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) = \bigcap_{\hat{a} \in \mathbf{a}^\forall} \bigcap_{\hat{b} \in \mathbf{b}^\forall} \bigcup_{\check{a} \in \mathbf{a}^\exists} \bigcup_{\check{b} \in \mathbf{b}^\exists} \{x \in \mathbb{R}^n \mid F(\hat{a} + \check{a}, x) = \hat{b} + \check{b}\}.$$

Proof. According to the definitions of intersection and union of sets

$$\begin{aligned} \Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) &= \{x \in \mathbb{R}^n \mid (\forall \hat{a} \in \mathbf{a}^\forall)(\forall \hat{b} \in \mathbf{b}^\forall)(\exists \check{a} \in \mathbf{a}^\exists)(\exists \check{b} \in \mathbf{b}^\exists) \\ &\qquad \big(F(\hat{a} + \check{a}, x) = \hat{b} + \check{b}\big)\} \\ &= \bigcap_{\hat{a} \in \mathbf{a}^\forall} \bigcap_{\hat{b} \in \mathbf{b}^\forall} \{x \in \mathbb{R}^n \mid (\exists \check{a} \in \mathbf{a}^\exists)(\exists \check{b} \in \mathbf{b}^\exists) \ \big(F(\hat{a} + \check{a}, x) = \hat{b} + \check{b}\big)\} \\ &= \bigcap_{\hat{a} \in \mathbf{a}^\forall} \bigcap_{\hat{b} \in \mathbf{b}^\forall} \left(\bigcup_{\check{a} \in \mathbf{a}^\exists} \bigcup_{\check{b} \in \mathbf{b}^\exists} \{x \in \mathbb{R}^n \mid F(\hat{a} + \check{a}, x) = \hat{b} + \check{b}\} \right). \end{aligned}$$

□

The value of the next result is mainly theoretical, but it facilitates better understanding the essense of the generalized solution sets by exposing their minimax nature.

THEOREM 3.2. *Let the mapping F be such that each controlling parameter $a_{k+1}, \dots, a_l$, which correspond to $\exists$-type of uncertainty, occurs in at most one of the components $F_i(a,x)$. Then the membership $x \in \Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ is equivalent to the following system of inequalities:*

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \min\limits_{\hat{a}\in\mathbf{a}^{\forall}} \max\limits_{\check{a}\in\mathbf{a}^{\exists}} F_i(\hat{a}+\check{a},x) \;\geq\; \overline{\mathbf{b}}_i, \\ \max\limits_{\hat{a}\in\mathbf{a}^{\forall}} \min\limits_{\check{a}\in\mathbf{a}^{\exists}} F_i(\hat{a}+\check{a},x) \;\leq\; \underline{\mathbf{b}}_i, \end{array}\right. \\ \qquad\qquad \text{—for the controlled outputs, } i = 1,\dots,s, \\ \left\{\begin{array}{l} \min\limits_{\hat{a}\in\mathbf{a}^{\forall}} \max\limits_{\check{a}\in\mathbf{a}^{\exists}} F_i(\hat{a}+\check{a},x) \;\geq\; \underline{\mathbf{b}}_i, \\ \max\limits_{\hat{a}\in\mathbf{a}^{\forall}} \min\limits_{\check{a}\in\mathbf{a}^{\exists}} F_i(\hat{a}+\check{a},x) \;\leq\; \overline{\mathbf{b}}_i, \end{array}\right. \\ \qquad\qquad \text{—for the stabilized outputs, } i = s+1,\dots,m. \end{array}\right. \tag{3.11}$$

Proof. Let $b = (b_1, b_2, \dots, b_m) = \hat{b} + \check{b}$, $\hat{b}, \check{b} \in \mathbb{R}^m$, $a = \hat{a} + \check{a}$, $\hat{a}, \check{a} \in \mathbb{R}^l$. We perform the following equivalent transformations with the separating predicate of the AE-solution set to the interval equation:

$$\begin{aligned} &\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) \\ &= \{x \in \mathbb{R}^n \mid (\forall \hat{a} \in \mathbf{a}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})(\exists \check{a} \in \mathbf{a}^{\exists})(\exists \check{b} \in \mathbf{b}^{\exists})\ \big(F(a,x) = b\big)\} \\ &= \{x \in \mathbb{R}^n \mid (\forall \hat{a} \in \mathbf{a}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})(\exists \check{a} \in \mathbf{a}^{\exists}) \\ &\qquad \big(F_1(a,x) = b_1 \quad \&\\ &\qquad\quad \cdots \quad \&\\ &\qquad F_s(a,x) = b_s \quad \&\\ &\qquad F_{s+1}(a,x) \in \mathbf{b}_{s+1} \quad \&\\ &\qquad\quad \cdots \quad \&\\ &\qquad F_m(a,x) \in \mathbf{b}_m\big)\} \\ &= \{x \in \mathbb{R}^n \mid (\forall \hat{a} \in \mathbf{a}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})(\exists \check{a} \in \mathbf{a}^{\exists}) \\ &\qquad \big(F_1(a,x) \geq b_1 \quad \& \quad F_1(a,x) \leq b_1 \quad \&\\ &\qquad\quad \cdots \quad \&\\ &\qquad F_s(a,x) \geq b_s \quad \& \quad F_s(a,x) \leq b_s \quad \&\\ &\qquad F_{s+1}(a,x) \geq \underline{\mathbf{b}}_{s+1} \quad \& \quad F_{s+1}(a,x) \leq \overline{\mathbf{b}}_{s+1} \quad \&\\ &\qquad\quad \cdots \quad \&\\ &\qquad F_m(a,x) \geq \underline{\mathbf{b}}_m \quad \& \quad F_m(a,x) \leq \overline{\mathbf{b}}_m\big)\} \end{aligned}$$

$$
\begin{aligned}
= \{x \in \mathbb{R}^n \mid\ & (\forall \hat{a} \in \mathbf{a}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall}) \\
& (\exists \check{a} \in \mathbf{a}^{\exists})(F_1(a,x) \geq b_1) \quad \& \quad (\exists \check{a} \in \mathbf{a}^{\exists})(F_1(a,x) \leq b_1) \quad \& \\
& \qquad \cdots \qquad \& \\
& (\exists \check{a} \in \mathbf{a}^{\exists})(F_s(a,x) \geq b_s) \quad \& \quad (\exists \check{a} \in \mathbf{a}^{\exists})(F_s(a,x) \leq b_s) \quad \& \\
& (\exists \check{a} \in \mathbf{a}^{\exists})(F_{s+1}(a,x) \geq \underline{\mathbf{b}}_{s+1}) \quad \& \quad (\exists \check{a} \in \mathbf{a}^{\exists})(F_{s+1}(a,x) \leq \overline{\mathbf{b}}_{s+1}) \quad \& \\
& \qquad \cdots \qquad \& \\
& (\exists \check{a} \in \mathbf{a}^{\exists})(F_m(a,x) \geq \underline{\mathbf{b}}_m) \quad \& \quad (\exists \check{a} \in \mathbf{a}^{\exists})(F_m(a,x) \leq \overline{\mathbf{b}}_m)\}.
\end{aligned}
$$

The last equality is true by virtue of the restriction we impose upon F: the sets of variables matching non-zero components of $\mathbf{a}^{\exists}$ that occur in different components of F simply do not intersect with each other, so we are allowed to "carry" the existential quantifiers to the separate members of the conjunctions [20], [50].

Notice that for functions f which are continuous over $\mathbf{a}$ we have the following equivalences:

$$(\exists a \in \mathbf{a})\ \big(f(a) \geq b\big) \quad \Longleftrightarrow \quad \max_{a \in \mathbf{a}} f(a) \geq b, \tag{3.12}$$

$$(\exists a \in \mathbf{a})\ \big(f(a) \leq b\big) \quad \Longleftrightarrow \quad \min_{a \in \mathbf{a}} f(a) \leq b. \tag{3.13}$$

Hence, we may continue our transformations as follows:

$$
\begin{aligned}
& \Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) \\
& = \Big\{ x \in \mathbb{R}^n \mid (\forall \hat{a} \in \mathbf{a}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall}) \\
& \qquad \Big(\Big(\max_{\check{a} \in \mathbf{a}^{\exists}} F_1(a,x) \geq b_1 \Big) \quad \& \quad \Big(\min_{\check{a} \in \mathbf{a}^{\exists}} F_1(a,x) \leq b_1 \Big) \quad \& \\
& \qquad\qquad \cdots \qquad \& \\
& \qquad \Big(\max_{\check{a} \in \mathbf{a}^{\exists}} F_s(a,x) \geq b_s \Big) \quad \& \quad \Big(\min_{\check{a} \in \mathbf{a}^{\exists}} F_s(a,x) \leq b_s \Big) \quad \& \\
& \qquad \Big(\max_{\check{a} \in \mathbf{a}^{\exists}} F_{s+1}(a,x) \geq \underline{\mathbf{b}}_{s+1} \Big) \quad \& \quad \Big(\min_{\check{a} \in \mathbf{a}^{\exists}} F_{s+1}(a,x) \leq \overline{\mathbf{b}}_{s+1} \Big) \quad \& \\
& \qquad\qquad \cdots \qquad \& \\
& \qquad \Big(\max_{\check{a} \in \mathbf{a}^{\exists}} F_m(a,x) \geq \underline{\mathbf{b}}_m \Big) \quad \& \quad \Big(\min_{\check{a} \in \mathbf{a}^{\exists}} F_m(a,x) \leq \overline{\mathbf{b}}_m \Big) \Big) \Big\}.
\end{aligned}
$$

Further,

$$(\forall\, b \in \mathbf{b})\ \big(f(a) \geq b\big) \quad \Longleftrightarrow \quad f(a) \geq \overline{\mathbf{b}},$$

$$(\forall\, b \in \mathbf{b})\ \big(f(a) \leq b\big) \quad \Longleftrightarrow \quad f(a) \leq \underline{\mathbf{b}},$$

so we have

$$
\begin{aligned}
&\Xi_{\alpha\beta}(F,\mathbf{a},\mathbf{b})\\
&=\Big\{x\in\mathbb{R}^n \mid (\forall \hat{a}\in\mathbf{a}^{\forall})\\
&\qquad\Big(\Big(\max_{\check{a}\in\mathbf{a}^{\exists}} F_1(a,x)\geq\overline{\mathbf{b}}_1\Big) \quad \&\quad \Big(\min_{\check{a}\in\mathbf{a}^{\exists}} F_1(a,x)\leq\underline{\mathbf{b}}_1\Big) \quad \&\\
&\qquad\qquad \dots \quad \&\\
&\qquad\Big(\max_{\check{a}\in\mathbf{a}^{\exists}} F_s(a,x)\geq\overline{\mathbf{b}}_s\Big) \quad \&\quad \Big(\min_{\check{a}\in\mathbf{a}^{\exists}} F_s(a,x)\leq\underline{\mathbf{b}}_s\Big) \quad \&\\
&\qquad\Big(\max_{\check{a}\in\mathbf{a}^{\exists}} F_{s+1}(a,x)\geq\underline{\mathbf{b}}_{s+1}\Big) \quad \&\quad \Big(\min_{\check{a}\in\mathbf{a}^{\exists}} F_{s+1}(a,x)\leq\overline{\mathbf{b}}_{s+1}\Big) \quad \&\\
&\qquad\qquad \dots \quad \&\\
&\qquad\Big(\max_{\check{a}\in\mathbf{a}^{\exists}} F_m(a,x)\geq\underline{\mathbf{b}}_m\Big) \quad \&\quad \Big(\min_{\check{a}\in\mathbf{a}^{\exists}} F_m(a,x)\leq\overline{\mathbf{b}}_m\Big)\Big)\Big\}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
(\forall\, a\in\mathbf{a})\ \big(f(a)\geq b\big) \quad &\Longleftrightarrow \quad \min_{a\in\mathbf{a}} f(a)\geq b,\\
(\forall\, a\in\mathbf{a})\ \big(f(a)\leq b\big) \quad &\Longleftrightarrow \quad \max_{a\in\mathbf{a}} f(a)\leq b,
\end{aligned}
$$

and we get

$$
\begin{aligned}
&\Xi_{\alpha\beta}(F,\mathbf{a},\mathbf{b})\\
&=\Big\{x\in\mathbb{R}^n \mid\\
&\qquad\Big(\min_{\hat{a}\in\mathbf{a}^{\forall}}\max_{\check{a}\in\mathbf{a}^{\exists}} F_1(a,x)\geq\overline{\mathbf{b}}_1\Big) \quad \&\quad \Big(\max_{\hat{a}\in\mathbf{a}^{\forall}}\min_{\check{a}\in\mathbf{a}^{\exists}} F_1(a,x)\leq\underline{\mathbf{b}}_1\Big) \quad \&\\
&\qquad\qquad \dots \quad \&\\
&\qquad\Big(\min_{\hat{a}\in\mathbf{a}^{\forall}}\max_{\check{a}\in\mathbf{a}^{\exists}} F_s(a,x)\geq\overline{\mathbf{b}}_s\Big) \quad \&\quad \Big(\max_{\hat{a}\in\mathbf{a}^{\forall}}\min_{\check{a}\in\mathbf{a}^{\exists}} F_s(a,x)\leq\underline{\mathbf{b}}_s\Big) \quad \&\\
&\qquad\Big(\min_{\hat{a}\in\mathbf{a}^{\forall}}\max_{\check{a}\in\mathbf{a}^{\exists}} F_{s+1}(a,x)\geq\underline{\mathbf{b}}_{s+1}\Big) \quad \&\quad \Big(\max_{\hat{a}\in\mathbf{a}^{\forall}}\min_{\check{a}\in\mathbf{a}^{\exists}} F_{s+1}(a,x)\leq\overline{\mathbf{b}}_{s+1}\Big) \quad \&\\
&\qquad\qquad \dots \quad \&\\
&\qquad\Big(\min_{\hat{a}\in\mathbf{a}^{\forall}}\max_{\check{a}\in\mathbf{a}^{\exists}} F_m(a,x)\geq\underline{\mathbf{b}}_m\Big) \quad \&\quad \Big(\max_{\hat{a}\in\mathbf{a}^{\forall}}\min_{\check{a}\in\mathbf{a}^{\exists}} F_m(a,x)\leq\overline{\mathbf{b}}_m\Big)\Big\},
\end{aligned}
$$

which coincides with the system (3.11). □

3.3. Quantifier Formalism in the Linear Case

In the rest of our paper, we consider more thoroughly the simplest interval *linear* algebraic systems (ILAS)

$$\mathbf{A}x=\mathbf{b} \tag{3.14}$$

with an interval $m \times n$-matrix $\mathbf{A} = (\mathbf{a}_{ij})$ and interval m-vector $\mathbf{b} = (\mathbf{b}_i)$. This kind of problem naturally arises, for instance, in the following situation.

Let us be given a static system of the form described in Section 2, about which the following assumptions are made:

(i) all the components $F_i(a, x)$ are bilinear functions of x and a, that is,

$$F_i(a, x) = \sum_{j,k} h_{ijk} a_j x_k$$

with some known coefficients $h_{ijk} \in \mathbb{R}$,

(ii) each of a_j occurs only once (if at all) in at most one of the component expressions $F_i(a, x)$ of the above bilinear form.

The latter means, in particular, that each of the system inputs a_j, $j = 1, 2, \ldots, l$, may affect only one of the outputs F_i, $i = 1, 2, \ldots, m$. In these conditions, the index k becomes unnecessary, but it makes sense to introduce an additional index i for the inputs a_j's indicating the component F_i in which they occur. Then a_j's turn into a_{ij}'s and one can put, without loss of generality, that all the scaling coefficients h_{ij} are equal to 1. Overall, we take

$$F_i(a, x) = \sum_j a_{ij} x_j, \qquad i = 1, 2, \ldots, m, \tag{3.15}$$

or, which is equivalent,

$$F(a, x) = Ax \tag{3.16}$$

with an $m \times n$-matrix $A = (a_{ij})$. Accordingly, if the interval uncertainty is present in the system inputs and/or outputs, we arrive at an interval linear system of the form (3.14). In practice, expressions (3.15)–(3.16) may occur, for example, in linear decision models of the form

$$\sum_{k=1}^{K} w_k a_{ik},$$

where w_k are criteria weights and a_{ik} may represent either partial utilities in multiattributive utility theory [46] or local priorities of alternatives in the Analytic Hierarchy Process [94] or something else.

We reformulate the notions and concepts advanced in Section 3.1 to fit the features of the interval linear systems of the above form.

DEFINITION 3.3. *AE-solution sets* are generalized solution sets to interval linear systems for which the separating predicate has AE-form.

As in the general case treated in Section 3.1, there are three equivalent ways to describe which uncertainty type is represented by this or that interval parameter of the system:

1. The first way is direct pointing of the quantifiers which are applied in particular intervals. Let the entire set of the index pairs (i,j) of the elements a_{ij}, that is, the set

   $$\{(1,1),(1,2),\ldots,(1,n),(2,1),(2,2),\ldots,(2,n),\ldots,(m,1),(m,2),\ldots,(m,n)\},$$

   be divided into two nonintersecting parts

   $$\hat{\Gamma} = \{\hat{\gamma}_1,\ldots,\hat{\gamma}_p\} \subset \mathbb{N}^2 \qquad \text{and} \qquad \check{\Gamma} := \{\check{\gamma}_1,\ldots,\check{\gamma}_q\} \subset \mathbb{N}^2,$$

   $p+q=mn$, such that

   - the parameter a_{ij} belongs to A-type of the interval uncertainty for $(i,j) \in \hat{\Gamma}$,
   - the parameter a_{ij} belongs to E-type of the interval uncertainty for $(i,j) \in \check{\Gamma}$.

   Similarly, we introduce nonintersecting sets of integer indices

   $$\hat{\Delta} = \{\hat{\delta}_1,\ldots,\hat{\delta}_s\} \subset \mathbb{N} \qquad \text{and} \qquad \check{\Delta} = \{\check{\delta}_1,\ldots,\check{\delta}_t\} \subset \mathbb{N},$$

   $s+t=m$, such that, in the right-hand side,

   - the element b_i is subsumed under the interval A-uncertainty for $i \in \hat{\Delta}$,
   - the element b_i is subsumed under the interval E-uncertainty for $i \in \check{\Delta}$.

   Also, we allow the natural possibility for some of the sets $\hat{\Gamma}$, $\check{\Gamma}$, $\hat{\Delta}$, $\check{\Delta}$ to be empty.

2. Like for the general interval equations of the form (1.1), to visually represent various uncertainty types that correspond to the elements of the linear system, it is convenient to introduce the quantifier $m \times n$-matrix $\alpha = (\alpha_{ij})$ and m-vector $\beta = (\beta_i)$ such that

   $$\alpha_{ij} = \begin{cases} \forall, & \text{if } (i,j) \in \hat{\Gamma}, \\ \exists, & \text{if } (i,j) \in \check{\Gamma}, \end{cases} \qquad \beta_i = \begin{cases} \forall, & \text{if } i \in \hat{\Delta}, \\ \exists, & \text{if } i \in \check{\Delta}. \end{cases} \tag{3.17}$$

3. The third way to specify the uncertainty types distribution for an interval linear system is to determine disjoint decompositions of both the interval matrix of the system and its right-hand side. As before, we define interval matrices $\mathbf{A}^{\forall} = (\mathbf{a}_{ij}^{\forall})$ and $\mathbf{A}^{\exists} = (\mathbf{a}_{ij}^{\exists})$ and interval vectors $\mathbf{b}^{\forall} = (\mathbf{b}_i^{\forall})$ and $\mathbf{b}^{\exists} = (\mathbf{b}_i^{\exists})$ of the same sizes as $\mathbf{A}$ and $\mathbf{b}$ as follows:

   $$\mathbf{a}_{ij}^{\forall} = \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{a}_{ij}^{\exists} = \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \exists, \\ 0, & \text{otherwise}, \end{cases} \tag{3.18}$$

   $$\mathbf{b}_i^{\forall} = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{b}_i^{\exists} = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \exists, \\ 0, & \text{otherwise}. \end{cases} \tag{3.19}$$

   Thus

   $$\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}, \qquad \mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists},$$
   $$\mathbf{a}_{ij}^{\forall} \cdot \mathbf{a}_{ij}^{\exists} = 0, \qquad \mathbf{b}_i^{\forall} \cdot \mathbf{b}_i^{\exists} = 0$$

for all i, j, that is, the matrices $\mathbf{A}^{\forall}$, $\mathbf{A}^{\exists}$ and vectors $\mathbf{b}^{\forall}$, $\mathbf{b}^{\exists}$ really form disjoint decompositions for $\mathbf{A}$ and $\mathbf{b}$ respectively.

DEFINITION 3.4. Let the uncertainty types distribution over the elements of the matrix $\mathbf{A}$ and right-hand side $\mathbf{b}$ be described by the quantifier $m \times n$-matrix $\alpha = (\alpha_{ij})$ and m-vector $\beta = (\beta_i)$ (defined by (3.17)), or by the equivalent decomposition of the index sets to subsets $\hat{\Gamma}$, $\check{\Gamma}$, $\hat{\Delta}$, $\check{\Delta}$. We will refer to the set

$$\begin{aligned}
&\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \\
&= \{x \in \mathbb{R}^n \mid \\
&\qquad (\forall a_{\hat{\gamma}_1} \in \mathbf{a}_{\hat{\gamma}_1}) \ldots (\forall a_{\hat{\gamma}_p} \in \mathbf{a}_{\hat{\gamma}_p})\, (\forall b_{\hat{\delta}_1} \in \mathbf{b}_{\hat{\delta}_1}) \ldots (\forall b_{\hat{\delta}_s} \in \mathbf{b}_{\hat{\delta}_s}) \\
&\qquad (\exists a_{\check{\gamma}_1} \in \mathbf{a}_{\check{\gamma}_1}) \ldots (\exists a_{\check{\gamma}_q} \in \mathbf{a}_{\check{\gamma}_q})\, (\exists b_{\check{\delta}_1} \in \mathbf{b}_{\check{\delta}_1}) \ldots (\exists b_{\check{\delta}_t} \in \mathbf{b}_{\check{\delta}_t}) \\
&\qquad\qquad (Ax = b)\},
\end{aligned} \tag{3.20}$$

as *AE-solution set of the type* $\alpha\beta$ to the interval linear system $\mathbf{A}x = \mathbf{b}$.

As before, the following well-known solution sets to interval linear systems—

- the *united solution set* (often called simply *solution set*, see, e.g., [2], [32], [45], [68], [69] and extensive references there)

$$\Xi_{uni}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists\, b \in \mathbf{b})\ \ (Ax = b)\},$$

- *tolerable solution set* (see, e.g., [48], [69], [119], [120])

$$\Xi_{tol}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists\, b \in \mathbf{b})\ \ (Ax = b)\},$$

- *controllable solution set* (see, e.g., [108], [114])

$$\Xi_{ctr}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists A \in \mathbf{A})\ \ (Ax = b)\}.$$

—are extreme points of a large family of $2^{m(n+1)}$ all possible AE-solution sets, i.e. having the form (3.20). The fourth extreme point of the family is the set

$$\{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\forall b \in \mathbf{b})\ \ (Ax = b)\}.$$

Considering it is not senseless, although mainly dull, since for the equations this solution set is empty in most cases.

In general terms, let the ith row of the matrix α entirely consists of the quantifiers $\forall$ and the respective element of the vector β is $\forall$ too. Then $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) = \emptyset$, if there is at least one interval with nonzero width amongst the elements $\mathbf{a}_{1j}, \ldots, \mathbf{a}_{in}, \mathbf{b}_i$. Because of this,

$$\binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m} = 2^m - 1$$

of the AE-solution sets prove *a priori* empty for the interval linear $m \times n$-system with the essentially interval elements. Overall, the number of "nontrivial" AE-solution sets lessens to $2^{m(n+1)} - 2^m + 1 = 2^m(2^{mn} - 1) + 1$ for such systems.

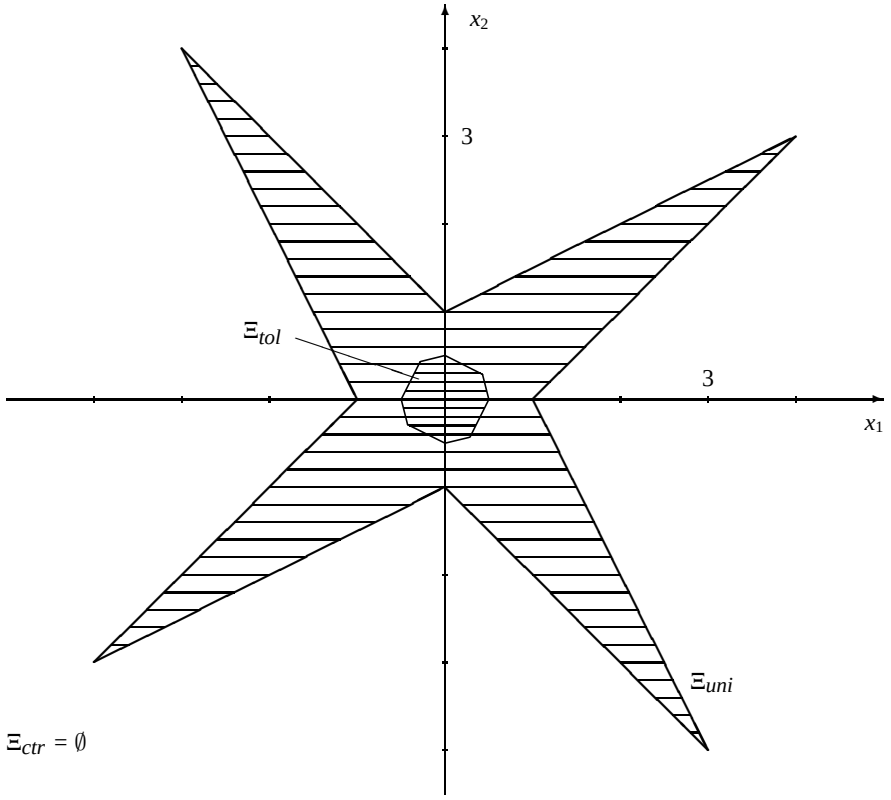

Figure 3. United solution set Ξ_{uni} and tolerable solution set Ξ_{tol} of the system (3.21).

For example, we can consider $2^2(2^4 - 1) + 1 = 61$ generalized AE-solution sets for an interval linear 2×2-system. Figures 3 and 4 show some solution sets to the popular interval linear system

$$\begin{pmatrix} [\ \ 2,4] & [-2,1] \\ [-1,2] & [\ \ 2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \tag{3.21}$$

from [6] repeatedly considered by many authors.

Notice that always $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) \subseteq \Xi_{uni}(\mathbf{A},\mathbf{b})$, that is, the united solution set is the widest in the collection of all AE-solution sets to interval systems. This observation can be generalized. Namely, if on the set of the logical quantifiers $\{\forall,\exists\}$ a partial ordering "$\preceq$" is introduced, setting $\forall \preceq \exists$, and the relationships $\alpha \preceq \alpha'$, $\beta \preceq \beta'$, $\alpha\beta \preceq \alpha'\beta'$ are understood componentwise and elementwise, then for any $\mathbf{A}$ and $\mathbf{b}$ there holds

$$\alpha\beta \preceq \alpha'\beta' \quad \Rightarrow \quad \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) \subseteq \Xi_{\alpha'\beta'}(\mathbf{A},\mathbf{b}). \tag{3.22}$$

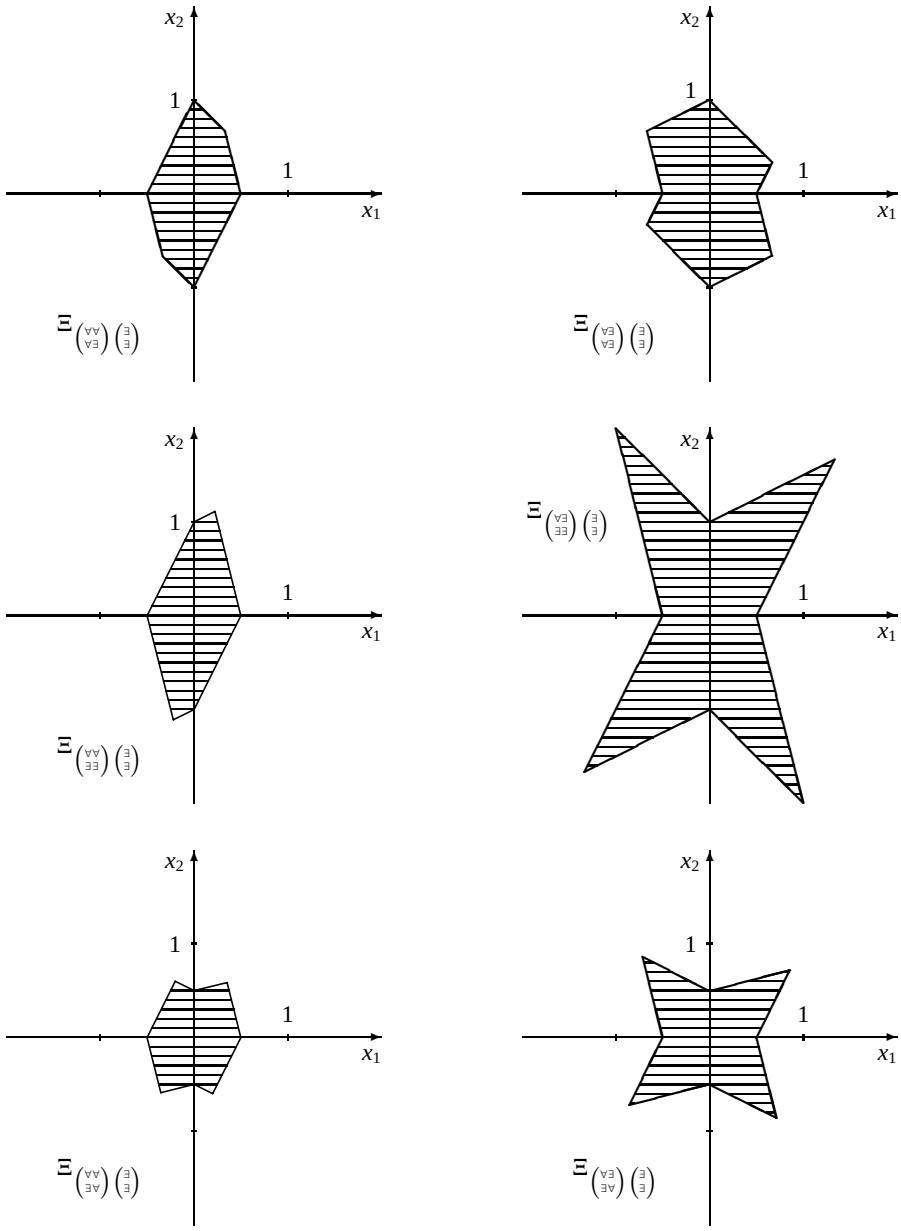

Figure 4. Some AE-solution sets of the interval linear system (3.21).

One can easily see that in Figures 3 and 4.

The property (3.22) may turn out very helpful when examining the generalized solution sets to interval equations systems. If we have already found out, for

example, that for the system (3.21)

$$\Xi_{\left(\begin{smallmatrix}\exists\\ \exists\end{smallmatrix}\right)\left(\begin{smallmatrix}\forall\\ \exists\end{smallmatrix}\right)} = \Xi_{\left(\begin{smallmatrix}\exists\\ \forall\end{smallmatrix}\right)\left(\begin{smallmatrix}\exists\\ \forall\end{smallmatrix}\right)} = \emptyset,$$

then, through "weakening" of quantifiers in the separating predicate, we can conclude that the controllable solution set Ξ_{ctr} for (3.21) is also empty, and such are another 45 solution sets to the system (3.21) that may be derived from the above three by combining the quantifiers before the elements of the matrix. The reasoning behind the property (3.22) may be equally extended to general interval nonlinear systems as well and, in fact, we have already done that in Theorems 6.4.

3.4. AE-Solution Sets for Interval Linear Systems

In this subsection, we derive various equivalent characterizations (descriptions) of the generalized AE-solution sets to interval linear systems and, relying on them, study some simple properties of these solution sets.

Prior to getting started, it is worth noting that the interval linear system of the form (3.14) is quite special in the sense that all its equations are independent from each other, and every interval parameter of the system occurs in only one equation. To exclude from our consideration the degenerate situations when all the interval parameters of a separate equation of (3.14) have A-uncertainty, we will take everywhere in the rest of the paper that, *for each* $i = 1, 2, \ldots, m$, *there exists at least one E-uncertain interval parameter among the entries*

$$\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{in}, \mathbf{b}_i$$

of the system (3.14) under study.

We give an obvious set-theoretical description of the AE-solution sets first.

THEOREM 3.3.

$$\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) = \bigcap_{\hat{a}\in\mathbf{a}^{\forall}} \bigcap_{\hat{b}\in\mathbf{b}^{\forall}} \bigcup_{\check{a}\in\mathbf{a}^{\exists}} \bigcup_{\check{b}\in\mathbf{b}^{\exists}} \{x\in\mathbb{R}^n \mid (\hat{A}+\check{A})\,x = \hat{b}+\check{b}\}.$$

In particular, if $\mathbf{A}$ *is a square nonsingular interval matrix, then*

$$\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) = \bigcap_{\hat{a}\in\mathbf{a}^{\forall}} \bigcap_{\hat{b}\in\mathbf{b}^{\forall}} \bigcup_{\check{a}\in\mathbf{a}^{\exists}} \bigcup_{\check{b}\in\mathbf{b}^{\exists}} (\hat{A}+\check{A})^{-1}(\hat{b}+\check{b}).$$

Proof. According to the definitions of intersection and unity of sets

$$\begin{aligned}\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) &= \{x\in\mathbb{R}^n \mid (\forall\hat{A}\in\mathbf{A}^{\forall})(\forall\hat{b}\in\mathbf{b}^{\forall})(\exists\check{A}\in\mathbf{A}^{\exists})(\exists\check{b}\in\mathbf{b}^{\exists})\\ &\qquad\qquad ((\hat{A}+\check{A})\,x = \hat{b}+\check{b})\}\\ &= \bigcap_{\hat{a}\in\mathbf{a}^{\forall}} \bigcap_{\hat{b}\in\mathbf{b}^{\forall}} \{x\in\mathbb{R}^n \mid (\exists\check{A}\in\mathbf{A}^{\exists})(\exists\check{b}\in\mathbf{b}^{\exists})\ ((\hat{A}+\check{A})\,x = \hat{b}+\check{b})\}\\ &= \bigcap_{\hat{a}\in\mathbf{a}^{\forall}} \bigcap_{\hat{b}\in\mathbf{b}^{\forall}} \bigcup_{\check{a}\in\mathbf{a}^{\exists}} \bigcup_{\check{b}\in\mathbf{b}^{\exists}} \{x\in\mathbb{R}^n \mid (\hat{A}+\check{A})\,x = \hat{b}+\check{b}\}. \qquad \square\end{aligned}$$

In particular, for the united solution set of the interval linear system (3.14) with the matrix $\mathbf{A}$ we have

$$\Xi_{uni}(\mathbf{A},\mathbf{b}) = \bigcup_{A\in\mathbf{A}} \bigcup_{b\in\mathbf{b}} \{x \in \mathbb{R}^n \mid Ax = b\},$$

which explains its name.

We turn now to the analytical characterizations of the generalized AE-solution sets to interval linear systems of the form (3.14). The fundamental result of our theory is

THEOREM 3.4. *Let, in the interval linear system* $\mathbf{A}x = \mathbf{b}$ *with* $m \times n$*-matrix* $\mathbf{A} = (\mathbf{a}_{ij})$ *and* m*-vector* $\mathbf{b} = (\mathbf{b}_i)$*, at least one of the interval elements* $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{in}, \mathbf{b}_i$ *is of E-uncertainty for every* $i = 1, 2, \ldots, m$*. Then the point* x *belongs to the solution set* $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ *if and only if*

$$\mathbf{A}^{\forall} \cdot x - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists} \cdot x, \tag{3.23}$$

where "$\cdot$" is interval matrix multiplication.

Proof. Using the matrices $\mathbf{A}^{\forall}$, $\mathbf{A}^{\exists}$ and vectors $\mathbf{b}^{\forall}$, $\mathbf{b}^{\exists}$ introduced by (3.18)–(3.19), we can rewrite Definition 3.4 of the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ in the following equivalent form:

$$\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall \hat{A} \in \mathbf{A}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})(\exists \check{A} \in \mathbf{A}^{\exists})(\exists \check{b} \in \mathbf{b}^{\exists}) \big((\hat{A}+\check{A})x = (\hat{b}+\check{b})\big)\}.$$

It is not hard to complete the proof of the theorem now, transforming equivalently the separating predicate of the solution set. We have

$$\begin{aligned}\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) &= \{x \in \mathbb{R}^n \mid (\forall \hat{A} \in \mathbf{A}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})(\exists \check{A} \in \mathbf{A}^{\exists})(\exists \check{b} \in \mathbf{b}^{\exists}) \\ &\qquad (\hat{A}x - \hat{b} = \check{b} - \check{A}x)\} \\ &= \{x \in \mathbb{R}^n \mid (\forall \hat{A} \in \mathbf{A}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})\ (\hat{A}x - \hat{b} \in \mathbf{b}^{\exists} - \mathbf{A}^{\exists} \cdot x)\} \\ &= \{x \in \mathbb{R}^n \mid \mathbf{A}^{\forall} \cdot x - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists} \cdot x\},\end{aligned}$$

since

$$\mathbf{b}^{\exists} - \mathbf{A}^{\exists} \cdot x = \{\check{b} - \check{A}x \mid \check{A} \in \mathbf{A}^{\exists},\ \check{b} \in \mathbf{b}^{\exists}\}$$

and

$$\mathbf{A}^{\forall} \cdot x - \mathbf{b}^{\forall} = \{\hat{A}x - \hat{b} \mid \hat{A} \in \mathbf{A}^{\forall},\ \hat{b} \in \mathbf{b}^{\forall}\}$$

in view of the properties of interval matrix operations [2], [69]. □

The above result was first obtained by S. Shary [107], [112]. It is pertinent to note that Theorem 3.4 generalizes all previously known characterizations of

various solution sets to interval linear systems—for the united solution set (Beeck's characterization, see [69]), for the tolerable solution set (see, e.g., [69], [71], [119]) and controllable solution set [59], [108], [114].

The next remarkable characterization result, which is due to J. Rohn [89], reformulates Theorem 3.4 in an analytical form using linear inequalities system with moduli thus generalizing the well-known Oettli-Prager theorem for the united solution set [76] and analogous characterization for the tolerable solution set [88]:

THEOREM 3.5. *The point x belongs to the solution set* $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ *if and only if*

$$|(\operatorname{mid}\mathbf{A})\cdot x-\operatorname{mid}\mathbf{b}|\le(\operatorname{rad}\mathbf{A}^{\exists}-\operatorname{rad}\mathbf{A}^{\forall})\cdot|x|+(\operatorname{rad}\mathbf{b}^{\exists}-\operatorname{rad}\mathbf{b}^{\forall}). \tag{3.24}$$

Proof. The inclusion $\mathbf{p}\subseteq\mathbf{q}$ of interval vectors $\mathbf{p},\mathbf{q}\in\mathbb{IR}^n$ is known (see, e.g., [69]) to be equivalent to the inequality

$$|\mathbf{q}-\operatorname{mid}\mathbf{p}|\le\operatorname{rad}\mathbf{q}-\operatorname{rad}\mathbf{p}.$$

Hence, the characterization (3.23) can be rewritten in the following form:

$$\begin{aligned}|(\mathbf{b}^{\exists}-\mathbf{A}^{\exists}\cdot x)-\operatorname{mid}(\mathbf{A}^{\forall}\cdot x-\mathbf{b}^{\forall})|&\\ \le\operatorname{rad}(\mathbf{b}^{\exists}-\mathbf{A}^{\exists}\cdot x)-\operatorname{rad}(\mathbf{A}^{\forall}\cdot x-\mathbf{b}^{\forall}).&\end{aligned} \tag{3.25}$$

Further,

$$\begin{aligned}\operatorname{rad}(\mathbf{p}\pm\mathbf{q})&=\operatorname{rad}\mathbf{p}+\operatorname{rad}\mathbf{q},\\ \operatorname{mid}(\mathbf{p}\pm\mathbf{q})&=\operatorname{mid}\mathbf{p}\pm\operatorname{mid}\mathbf{q}.\end{aligned}$$

Therefore, (3.25) holds if and only if

$$\begin{aligned}|\mathbf{b}^{\exists}-\operatorname{mid}(\mathbf{A}^{\exists}\cdot x)-\operatorname{mid}(\mathbf{A}^{\forall}\cdot x)+\operatorname{mid}\mathbf{b}^{\forall}|&\\ \le\operatorname{rad}\mathbf{b}^{\exists}+\operatorname{rad}(\mathbf{A}^{\exists}\cdot x)-\operatorname{rad}(\mathbf{A}^{\forall}\cdot x)-\operatorname{rad}\mathbf{b}^{\forall},&\end{aligned}$$

which is equivalent to Rohn's characterization (3.24) insofar as

$$\operatorname{mid}(\mathbf{A}^{\exists}\cdot x)=(\operatorname{mid}\mathbf{A}^{\exists})\cdot x,\qquad\operatorname{mid}(\mathbf{A}^{\forall}\cdot x)=(\operatorname{mid}\mathbf{A}^{\forall})\cdot x$$

and

$$\operatorname{rad}(\mathbf{A}^{\exists}\cdot x)=(\operatorname{rad}\mathbf{A}^{\exists})\cdot|x|,\qquad\operatorname{rad}(\mathbf{A}^{\forall}\cdot x)=(\operatorname{rad}\mathbf{A}^{\forall})\cdot|x|.$$ □

DEFINITION 3.5. The *vertices* of an interval vector $\mathbf{x}\in\mathbb{IR}^n$ are the vectors of the set

$$\operatorname{vert}\mathbf{x}:=\Big\{x\in\mathbb{R}^n\mid x_i\in\{\underline{\mathbf{x}}_i,\overline{\mathbf{x}}_i\},\ i=1,2,\ldots,n\Big\}.$$

Vertices of an interval matrix are defined in the similar way.

THEOREM 3.6. *For any quantifiers α and β, the intersection of the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ with each orthant of the space $\mathbb{R}^n$ is a convex polyhedral set whose vertices are the solutions of the extreme point linear systems $Ax = b$ with $A \in \operatorname{vert}\mathbf{A}$ and $b \in \operatorname{vert}\mathbf{b}$.*

Proof. Membership of a real vector x in some orthant is determined by fixing the signs of its components. Notice also that for any interval $m \times n$-matrix $\mathbf{C}$, the components of the product $\mathbf{C}\cdot x = ((\mathbf{C}\cdot x)_1, (\mathbf{C}\cdot x)_2, \ldots, (\mathbf{C}\cdot x)_m)^\top$ may be represented as follows:

$$(\mathbf{C}\cdot x)_i = \sum_{j=1}^{n} \mathbf{c}_{ij}x_j = \left[\sum_{j=1}^{n} \underline{\mathbf{c}_{ij}x_j},\ \sum_{j=1}^{n} \overline{\mathbf{c}_{ij}x_j}\right] = \left[\sum_{j=1}^{n} c'_{ij}x_j,\ \sum_{j=1}^{n} c''_{ij}x_j\right], \tag{3.26}$$

where c'_{ij} and c''_{ij} are some numbers (they may coincide) that belong to the set of endpoints $\{\underline{\mathbf{c}}_{ij}, \overline{\mathbf{c}}_{ij}\}$ and are fixed for any separate orthant containing x.

Next, writing out the inclusion (3.23) in componentwise manner and changing, on the basis of the representation (3.26), each one-dimensional inclusion by a pair of inequalities between the endpoints of the intervals, we get a system of $3n$ linear inequalities

$$\begin{cases} A'x \ \geq\ b', \\ A''x \leq b'', \\ \text{conditions on the signs of } x_i,\ i = 1, 2, \ldots, n, \end{cases} \tag{3.27}$$

where $A', A'' \in \operatorname{vert}\mathbf{A}$ and $b', b'' \in \operatorname{vert}\mathbf{b}$. The inequality system (3.27) determines a convex polyhedral set. □

Hence, $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ can be represented in general as the union of at most 2^n convex polyhedral sets—the fact widely known for the united solution set since the work by W. Oettli [75]. Anyway the complexity of the direct description of the solution sets $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ may thus grow exponentially with n despite the above simple and geometrically vivid characterization results. So, such a description turns out to be extremely laborious and practically useless even for moderate dimensions of interval equations. A. Lakeyev has shown [59] that even the problems of recognition of whether the united solution set $\Xi_{uni}(\mathbf{A},\mathbf{b})$ or controllable solution set $\Xi_{ctr}(\mathbf{A},\mathbf{b})$ is empty or not are NP-complete, that is, in general they can hardly be solved easier than by the time which is the exponential function of the length of their code [29].

Because of the above, it makes good sense not to aim at finding the complete description of $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ in practice, and the conclusion has, in fact, quite general character, being valid for a large class of the interval problem statement. In particular, we would suffice to compute only a simply constructed approximations of $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$, and the specific problem statements will be discussed at length in the next Section 4.

PROPOSITION 3.1. *For any quantifiers α and β, both the minimal and maximal componentwise values of the points from the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$, that is,*

$$\min\{x_\nu \mid x \in \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})\} \qquad \textit{and} \qquad \max\{x_\nu \mid x \in \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})\},$$

$\nu = 1, 2, \ldots, n$, are attained at the solutions of the extreme point linear systems $Ax = b$ with $A \in \operatorname{vert}\mathbf{A}$ and $b \in \operatorname{vert}\mathbf{b}$.

This is a straightforward consequence of Theorem 3.6. For the united solution set, the result was obtained by K. Nickel in [73] in a way different from ours.

4. Posing Interval Problems

4.1. DISCUSSION

Now that we have defined what the solution sets to interval systems of equations (inequalities, etc.) are, it is time to decide what to do with them further.

This is not an idle question. The fundamental fact about our reality is that we can observe, use and operate objects and concepts that are not just *finite*, but *not too complex* as well. The finiteness of our perception, considerations, calculations, etc., is widely recognized by the people and does not give rise to doubt. But what is so special with the interval problems that we have to impose the second requirement, "not too complex"? The answer is that, entering the realm of sets taken as essentially compound objects, made up of elementary parts, we encounter the growing (and even dominating) role of the combinatorial effects, which hardly reveal themselves in the usual point mathematics. The combinatorics is known to be the main source of very large and even huge numbers that can exceed any real physical quantities.

In particular, all we have said is true in full measure for generalized solution sets we introduces in the preceding section. Even in simple practical situations a direct computation and description of generalized solution sets prove, as a rule, arduous and sometimes almost impossible. For instance, the length of the exact description of the AE-solution sets generally grows faster than 2^n (the number of orthants of $\mathbb{R}^n$) in the linear case, when $F(a,x) = Ax$ with some $m \times n$-matrix A (see Section 3.4). So, such a description becomes larger than the famous "chess number" $2^{64} - 1$ even when the dimension of the problem is equal to mere 64. When the dimension reaches some hundreds, the number of orthants we have to list in the exact description of $\Xi_{\alpha\beta}$ becomes comparable with the number of elementary particles in the Universe.* One must be aware of the fact that the dimensions we treated in these examples are very moderate numbers. To compare, the modern economics involve thousands of participants, and even aggregated input-output models usually consider about a hundred of industries.

* For example, É. Borel in [13] pointed out 10^{200} as the maximal number of elementary events that might have took place in the Universe after the "Big Bang." This evaluation has not considerably increased since the time the book [13] was published.

Summing up, one would say that the solution sets to interval systems of equations have, in general, very large and even enormous *complexity* (often called *Kolmogorov complexity*, see [63]), which exponentially grows as the dimension of the system increases. This difficulty seems to be of principal character: the recent theoretical results by A. Lakeyev [56] show that the problems of recognition and estimation of the AE-solution sets to interval linear systems turn out to be intractable provided that we do not impose restrictions on the interval matrix of the system. See also fundamental survey [51].

The practical consequence of the above stated is an inevitable necessity to somehow coarsen the exact description of the solution sets, that is, to change the complete and too complex description of the solution set for a simpler set – its *approximation*,—with less description complexity, in a way that still meets the requirements of the practical problem under solution. Overall, such an approximation procedure should be incorporated into the problem statement while its details are to be scrutinized and consciously taken into account.

A specific approximation criterion and the employed approximating sets are to be determined for each particular problem. Besides, a measure (metric) ϱ is often drawn in the consideration to evaluate the deviation (in a prescribed sense) of the estimating set from the solution set. Finally, the distinguishing feature of the interval problems we deal with that makes them unlike, for instance, classical approximation problems, is the presence, apart from the metric ϱ, of an additional *qualitative* requirement that the answer to the problem must meet to, and that is not at all related to ϱ. This qualitative requirement is, as a rule, a condition for the mutual position (location) of the solution set and the estimating set. For example, when computing an enclosure of the range of values of a function, one does not simply need some estimates of the range, no matter how close they are, but only those evaluating the range *from below* and *from above* with guarantee. When the latter is not the case, the interval obtained must *not* be considered a solution to the range enclosing problem.

Further examples illustrating the peculiarity of the interval problems can be found in the next subsection.

4.2. What Is an Interval "Estimation Problem"?

Gathering up the threads of the previous discussion, we arrive at the following formalization of the concept of a class of interval analysis problems that we shall call *interval estimation problems*:

DEFINITION 4.1. A generic *interval estimation problem P* is an ordered quadruple of the form $(\mathcal{S}, \mathcal{E}, \mathcal{M}, \varrho)$, where

- $\mathcal{S}$ denotes a *family of solution sets*, that is, a mapping from an interval Π of $\mathbb{R}^r$ (or of a more general set) into a class of sets; Π describes possible values of the parameters of *P*, while an *individual interval estimation problem J* is extracted

from P by assigning to the variables of $\mathcal{S}$ specific values that determine (as a result of the solution process) an *individual solution set* $\mathfrak{S} \in \mathcal{S}$;

$\mathcal{E}$ designates a *class of estimating sets*, that is, a family of sets by which we are going to approximate the solution sets from $\mathcal{S}$;

$\mathcal{M}$ specifies a *way of estimation* of the solution set or, to put it otherwise, a binary relation between elements of $\mathcal{S}$ and elements of $\mathcal{E}$ that must be met according to the sense of the problem under study;

ϱ stands for a non-negative functional on $\mathcal{S} \times \mathcal{E}$ (*metric*) that indicates the "error" of the result, i.e., measures how close (in some sense) the estimating set is to the true solution set determined by the problem statement.

By a *solution* of the problem J we will understand an estimating set $\mathfrak{E} \in \mathcal{E}$ such that the relation $\mathfrak{S}\mathcal{M}\mathfrak{E}$ is satisfied and, optionally, a certain condition on the value $\varrho(\mathfrak{S}, \mathfrak{E})$ is met.

No doubt, one should clearly recognize that the interval analysis operates a number of problem statements that do not comply with the above scheme. These are, for instance, the problem of computing formal solutions to interval systems of equations (see the forthcoming sections), the problem of checking whether an interval matrix is regular or not, the problem of checking the stability (either by Hurwitz or Schur) of an interval matrix, and so on. Definition 4.1 draws apart a sufficiently wide and practically significant class of problems that interval analysis works with, so that it is quite natural to name it by a special term, and so did we.

In modern interval analysis, the most commonly encountered ways of approximating a solution set $\mathfrak{S}$ are known to be

- *outer interval estimation*, when we seek an interval vector $\mathbf{E}$ such that $\mathbf{E} \supseteq \mathfrak{S}$, and
- *inner interval estimation*, when we seek an interval vector $\mathbf{E}$ such that $\mathbf{E} \subseteq \mathfrak{S}$.

The above-described outer and inner estimations, while embracing a variety of widespread interval problems, are not at all the only possible ones. A lot of examples of other estimation modes could be given, and they cannot be treated only as a theoretical oddity.

Let, for instance, the estimating sets be n-dimensional interval vectors, $\mathcal{E} = \mathbb{IR}^n$. The outer interval estimation of a solution set $\mathfrak{S}$ by an interval $\mathbf{E}$ is evidently equivalent to

$$\mathrm{pr}_i \mathfrak{S} \subseteq \mathbf{E}_i, \qquad i = 1, 2, \ldots, n,$$

where pr_i is the operation of projecting onto the i-th coordinate axis. Requiring the inverse inclusions

$$\mathrm{pr}_i \mathfrak{S} \supseteq \mathbf{E}_i, \qquad i = 1, 2, \ldots, n,$$

for the estimating boxes, we get an example of non-traditional estimation mode, which can be called "weak inner estimation" (see Figure 5). This kind of estimation

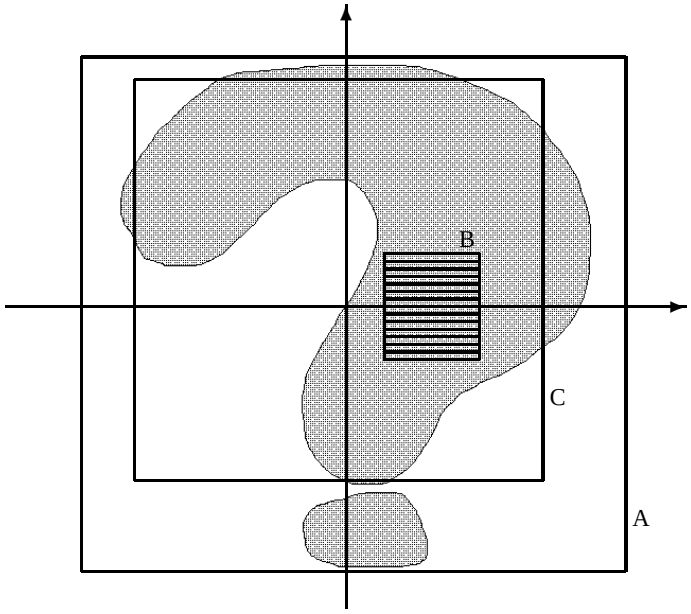

Figure 5. Estimation of the solution set (shaded question-mark) by "outer" box (A), "inner" box (B), and "weak inner" box (C).

is used in a number of identification problems [126], and when some one needs to know a sharpness (an amount of overestimation) in outer estimation problems [93].*

Sometimes, we must guarantee the enclosure $\mathrm{pr}_i \mathfrak{S} \subseteq \mathbf{E}_i$ only for some, not all, components $i \in \{1, 2, \ldots, n\}$, and for the other i the opposite inclusion $\mathrm{pr}_i \mathfrak{S} \supseteq \mathbf{E}_i$ is required. In other words, for some indexes i, one needs the lower estimate of $\min\{x_i \mid x \in \mathfrak{S}\}$, while for the rest i it is necessary to obtain the upper estimates of this value, the similar requirements being imposed on the estimation of $\max\{x_i \mid x \in \mathfrak{S}\}$. The estimation mode defined by the above conditions can be naturally termed "mixed estimation."

Now, let us turn to the classes of estimating sets. In practice, in addition to the common intervals, one-dimensional estimating sets can be represented by "intervals" of Kahan arithmetic [60], multiintervals (i.e., finite unions of intervals and infinite semi-lines [129]). In the multidimensional case, the variety of shapes of the solution sets complicates the situation: sometimes estimating by interval boxes, i.e. direct products of real intervals, may turn out inadequate, since provides us with little information, producing large overestimation and/or underestimation (see Figure 6). As a consequence, apart from the ordinary boxes, skew parallelotopes (zonotopes) and ellipsoids are commonly used as estimating sets (the latter being

* To specify this estimation mode, S. Rump uses in [93] the term "inner inclusion," which is, to our mind, not quite adequate to the situation.

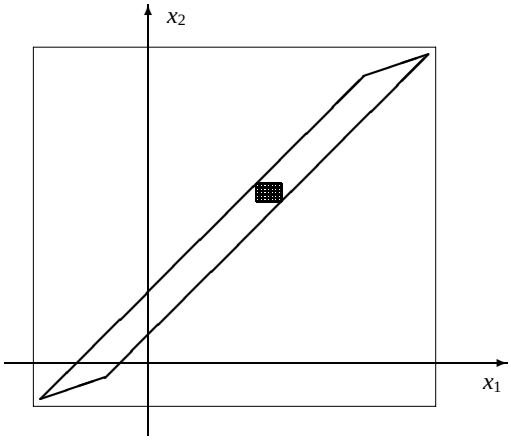

Figure 6. Sometimes interval estimation of the solution set (long skew parallelogram), either inner or outer, may turn out inadequate, since provides us with little information.

especially popular in solving ODEs; see [22], [15] and references there), balls with respect to a certain norm [125], intersections of several independently found parallelepipeds (which is taken at the final stage of the computational procedure) [19] and so on. Aside from the classical one-dimensional complex intervals such as rectangles and circles on a complex plane [2], circular rings [79] and circular sectors [49] et al. can be used. The evident conclusion which can be drawn from the above-mentioned examples is that there can be a variety of estimating sets and possible modes of estimation of the solution sets, which results in a great variety of interval problems statements.

Note the absolutely special role of the third member of the quadruple $(\mathcal{S}, \mathcal{E}, \mathcal{M}, \varrho)$, i.e., of the estimation mode $\mathcal{M}$. We can say that it is the presence of the relation $\mathcal{M}$ that primarily determines the peculiarity of the form of the interval problems. As we have already mentioned, the solution to such a problem must first of all satisfy some *qualitative condition* expressed by the estimation mode $\mathcal{M}$ and only afterward error, closeness, etc. are taken into account.

Summing up, one can say that the answer to the question "what to do with the solution set?" depends on the problem under consideration, i.e., on the final purposes of the system analysis as applied to our problem (2.2).

4.3. Problems To Be Considered

In the sequel, we are going to consider the two most popular interval problems—*inner* and *outer*, that is, estimation of the AE-solution sets to interval systems of equations by *subsets* and *supersets*.

Estimating by subsets is the only valid one if we are interested in getting a collection of points that provide us with right answers to the main question (2.2). In other words, only for subsets $\Pi \subseteq \Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ does our major characteristic

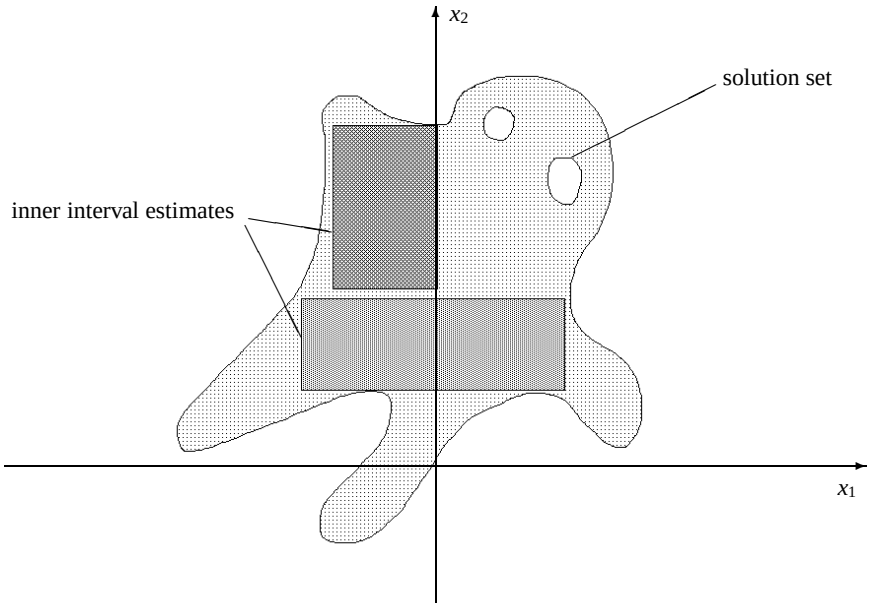

Figure 7. "Inner problems" are the problems of *inner* (interval) estimation of the solution sets.

property—the selecting predicate of the solution set—remain satisfied for all the points $x \in \Pi$, while the estimating sets being in any other relation with the solution set may contain points that have nothing to do with the answers to the question (2.2), which may turn out unacceptable in practice.

Taking the estimating subsets in the form of the axis-aligned boxes (interval vectors), we thus arrive at the problem of inner interval estimation of the solution sets (2.4):

$$\begin{aligned} &\textit{For an interval equation } F(\mathbf{a}, x) = \mathbf{b} \textit{ and quantifier vectors} \\ &\alpha \textit{ and } \beta \textit{ of the same size as } \mathbf{a} \textit{ and } \mathbf{b} \textit{ respectively, find} \\ &\textit{an inner interval estimate of the solution set } \Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}). \end{aligned} \tag{4.1}$$

Henceforward, we shall refer to this problem as the *inner problem* for the interval system $F(\mathbf{a}, x) = \mathbf{b}$ (as in Figure 7), pointing out, if necessary, that the distribution of interval uncertainty types over the system parameters is described by the quantifiers α and β, or, which is equivalent, by the disjoint decompositions $\mathbf{a} = \mathbf{a}^{\forall} + \mathbf{a}^{\exists}$ and $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$.

It is very useful to show practical meaning of the problem (4.1) on particular examples. If the tolerable solution set $\Xi_{tol}(F, \mathbf{a}, \mathbf{b})$ is taken as a case in point in the above definition, then the problem (4.1) is the classical *tolerance problem* [18], [48], [69], [119], [120], which has numerous and fruitful practical applications. The tolerance problem is actually the problem of stabilization of the system within

a prescribed output corridor $\mathbf{b}$ for the case when *all* the system parameters a_i are subject to some bounded disturbances.

If some a_i's have A-uncertainty while the rest of them have E-uncertainty, that is, some parameters are disturbing and some are controllable, and all $\beta_i = \exists$, $i = 1, 2, \ldots, m$, then we get the stabilization problem with a control possibility, which some of the researchers call "problem of ensuring the operation stability under large-scale disturbances" [130]. Such is the quality control problem from [30] which we considered in Section 3. In an important methodology paper [4], the "operation stability" problem statement has been illustrated on concrete practical examples from shipbuilding, toxicology, economics and electrical power engineering. The other title of this kind of problem also extensively used in the literature on the subject is "the problem of ensuring survival of the system" [4]. Alternately, if part of a_i's are A-parameters and a part of them are E-parameters while all $\beta_i = \forall$, $i = 1, 2, \ldots, m$, then we have the control problem under bounded perturbations.

Outer estimation of the generalized solution sets $\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ also makes sense in the sensitivity-like analysis of the systems, but it would be another form of the problem (2.2), different from (4.1):

> *For an interval equation* $F(\mathbf{a}, x) = \mathbf{b}$ *and quantifier vectors* α *and* β *of the same size as* $\mathbf{a}$ *and* $\mathbf{b}$ *respectively, find an outer interval estimate of the solution set* $\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$. (4.2)

Henceforward, we shall refer to this problem as the *outer problem* for the interval system $F(\mathbf{a}, x) = \mathbf{b}$ (as in Figure 8), pointing out, if necessary, that the distribution of interval uncertainty types over the system parameters is described by the quantifiers α and β, or, which is equivalent, by the disjoint decompositions $\mathbf{a} = \mathbf{a}^{\forall} + \mathbf{a}^{\exists}$ and $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$.

Since we are going to restrict ourselves mainly to the interval linear systems, the following specific problems will be our major concern in the next sections:

> *For an interval linear equation* $\mathbf{A}x = \mathbf{b}$ *and quantifier vectors* α *and* β *of the same size as* $\mathbf{a}$ *and* $\mathbf{b}$ *respectively, find an inner interval estimate of the solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. (4.3)

and

> *For an interval equation* $F(\mathbf{a}, x) = \mathbf{b}$ *and quantifier vectors* α *and* β *of the same size as* $\mathbf{a}$ *and* $\mathbf{b}$ *respectively, find an outer interval estimate of the solution set* $\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$. (4.4)

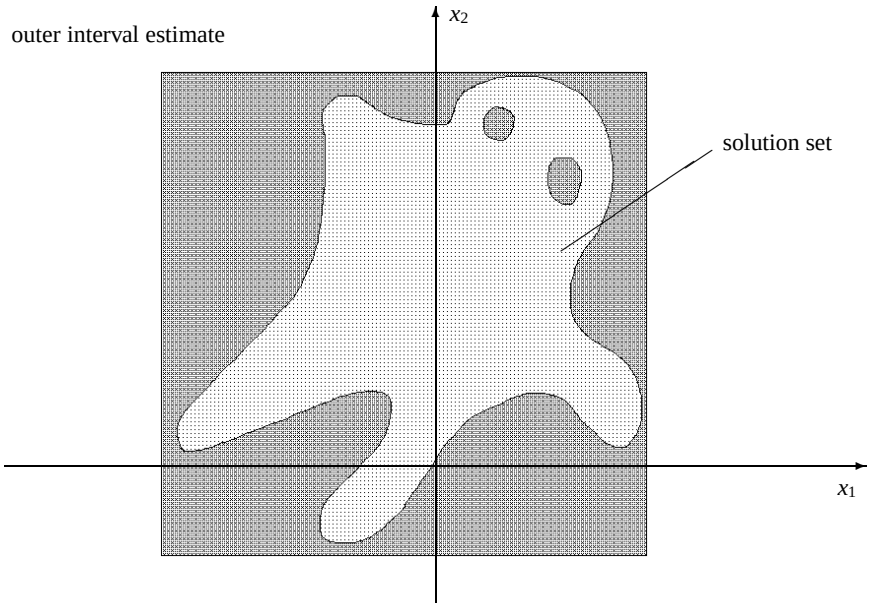

Figure 8. "Outer problems" are the problems of *outer* (interval) estimation of the solution sets.

Sometimes, it is more convenient to consider a componentwise form of the problem (4.4):

For an interval system of linear equations $\mathbf{A}x = \mathbf{b}$
find estimates for $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ *from below,*
for $\max\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ *from above,* $v = 1, 2, \ldots, n.$

(4.5)

Clearly these problem statements make sense only in the case $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \neq \emptyset$, and finding out the conditions of such nonemptyness is a separate important question.

So far, the problems analogous to (4.1)–(4.4) have been being solved only by minimax methods of mathematical programming (see, in particular, [4], [130]). One of the main purposes of our paper is to develop new computationally efficient interval approaches to the analysis of static systems under interval uncertainty, that is, to the solution of the problems (4.1) and (4.2). The principles that will underlie our techniques are somewhat uncommon to modern interval analysis. We develop, in particular, the so-called *formal approach* to the solution of the above mentioned problems, and the key concept in many of our consideration is that of *formal solution* to the interval equation (sometimes referred to as *algebraic solution*):

DEFINITION 4.2. An interval (interval vector, matrix, etc.) is called a *formal solution* to the interval equation (system of equations, inequalities, etc.) if substituting

this interval into the equation and executing all interval arithmetical, analytical, etc., operations result in a valid equality.

For instance, the interval $[0, 1]$ is the formal solution to the interval quadratic equation $[1, 2]x^2 + [-1, 1]x = [-1, 3]$. The interval function $\mathbf{x}(t) = 10.5[e^t, e^{2t}]$ is a formal solution to the interval differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = [1, 2].$$

The interval function $\mathbf{x}(t) = [0, 2t]$ on [0,1] is a formal solution to the following interval Fredholm integral equation of the second kind

$$x(t) + \int_0^1 (1.5s + t)\, x(s)\, \mathrm{d}s = [0, 3t + 1].$$

The latter (purely illustrative) examples show what is bad with the former term *algebraic solution*: it emphasizes only the algebraic nature of operations that compose the systems under consideration, so that speaking of "algebraic" solutions to interval differential, integral, etc., equations is at least strange and misleading.

The notion of formal solution thus corresponds to the usual concept of a solution to an equation, which is common to the most of mathematics in general. Such understanding of a solution to an interval equation was originated by S. Berti in [10], who gave no name to it and considered only one very simple equation. Afterward, K. Nickel [72] and H. Ratschek and W. Sauer [85] considered this kind of solution to interval linear equations and systems of such equations. H. Ratschek and W. Sauer introduced the currently popular term "algebraic solution."

The essence of our formal approach is to change the problems (4.1)–(4.5) to the problem of finding formal solutions to some special equations in *Kaucher complete interval arithmetic* $\mathbb{KR}$, thus reducing the initial problem to a traditional problem of numerical analysis. This reduction is a very attractive feature, notwithstanding that the formal solution to the auxiliary interval equation does not need to exist even if the corresponding original problem (4.1)–(4.5) has solutions, that is, the solution set is not empty.

5. Kaucher Complete Interval Arithmetic

The main tool of interval analysis is known to be the so-called *interval arithmetics*, i.e. algebraic structures that formalize arithmetical operations between intervals as entire undivided objects. Below, we give a short critical overview of the classical interval arithmetic and afterward present a description of *Kaucher complete interval arithmetic*, which plays one of the leads in our theory, but has not been sufficiently known to the researchers.

5.1. DISCUSSION

We have already seen in Theorem 3.2 that our main problems (2.2), (4.1), and (4.2) are *minimax* by their nature, i.e., such ones that involve taking minimaxes of a multivariable function. Then, to solve minimax problems we need a special "minimax" interval arithmetic, that is, an interval arithmetic implementing computation of minimax at each elementary operation, namely addition, subtraction, multiplication and division (which are multivariable functions as well). Classical interval arithmetic and its well-known generalizations—Kahan arithmetic [60], Hansen arithmetic (see [32]) and some others—are designed for evaluation of the range of arithmetical operations, or alternatively, to compute pure minima and maxima. Hence, these interval arithmetics are not suitable for our purpose.

Luckily, a "minimax" interval arithmetic does exist and we do not need to construct it by ourselves on a bare place. It is *Kaucher interval arithmetic*, also called *complete interval arithmetic*, which we have already mentioned earlier.

Classical interval arithmetic $\mathbb{IR}$ is known to be an algebraic system of all real intervals $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x \in \mathbb{R} \mid \underline{\mathbf{x}} \le x \le \overline{\mathbf{x}}\}$, with the binary arithmetical operations defined according to the following basic principle:

$$\mathbf{x} \star \mathbf{y} = \{x \star y \mid x \in \mathbf{x},\ y \in \mathbf{y}\} \tag{5.1}$$

for all the intervals $\mathbf{x}$, $\mathbf{y}$ such that $(x \star y)$, $\star \in \{+, -, \cdot, /\}$ makes sense for any $x \in \mathbf{x}$, $y \in \mathbf{y}$ (see, e.g., [2], [32], [45], [68], [69]). The explicit formulas of the interval arithmetical operations are

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &:= [\underline{\mathbf{x}} + \underline{\mathbf{y}},\ \overline{\mathbf{x}} + \overline{\mathbf{y}}],\\
\mathbf{x} - \mathbf{y} &:= [\underline{\mathbf{x}} - \overline{\mathbf{y}},\ \overline{\mathbf{x}} - \underline{\mathbf{y}}],\\
\mathbf{x} \cdot \mathbf{y} &:= [\min\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\},\ \max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\}],\\
\mathbf{x} / \mathbf{y} &:= \mathbf{x} \cdot [1 / \overline{\mathbf{y}},\ 1 / \underline{\mathbf{y}}] \qquad \text{for } \mathbf{y} \not\ni 0.
\end{aligned}$$

Algebraic properties of classical interval arithmetic are meager. It is not even a group [12] both with respect to addition and multiplication: intervals with nonzero width, that is, the majority of elements of $\mathbb{IR}$, do not have algebraic opposite and inverse ones (in the group sense). Besides, $\mathbb{IR}$ is not a lattice [12] with respect to the natural set-theoretical inclusion ordering "$\subseteq$". The first of the lattice operations

$$\mathbf{x} \wedge \mathbf{y} := \inf_{\subseteq}\{\mathbf{x}, \mathbf{y}\} = [\max\{\underline{\mathbf{x}}, \underline{\mathbf{y}}\},\ \min\{\overline{\mathbf{x}}, \overline{\mathbf{y}}\}], \tag{5.2}$$

—taking greatest lower bound with respect to $\subseteq$,—

$$\mathbf{x} \vee \mathbf{y} := \sup_{\subseteq}\{\mathbf{x}, \mathbf{y}\} = [\min\{\underline{\mathbf{x}}, \underline{\mathbf{y}}\},\ \max\{\overline{\mathbf{x}}, \overline{\mathbf{y}}\}], \tag{5.3}$$

—taking least upper bound with respect to $\subseteq$,—

is not always applicable in classical interval arithmetic. For example, $[1,2] \wedge [3,5]$ would have to be $[3,2]$ or, in the last resort, empty set, but neither "backward" intervals nor $\emptyset$ are allowed in $\mathbb{IR}$.

"Incompleteness" of both algebraic and order structures of $\mathbb{IR}$ naturally stimulated attempts to complete classical interval arithmetic, to create a "more convenient" interval arithmetic based on $\mathbb{IR}$, and the most successful of them was due to E. Kaucher back in 70's. The joint order-algebraic completion of $\mathbb{IR}$ carried out in the works by E. Kaucher [42]–[44] resulted in an algebraic system that we will call *complete interval arithmetic* $\mathbb{KR}$, or *Kaucher interval arithmetic*. Afterward, E. Gardeñes, M. Sainz and their co-workers studied this arithmetic (which they called *arithmetic of modal intervals*) and established some of its helpful properties, applications as well as having implemented them in workable computer systems [24]–[28].

5.2. Description of the Arithmetic

Elements of $\mathbb{KR}$ are pairs of real numbers $[\underline{x}, \overline{x}]$ that are not connected by the obligatory condition $\underline{x} \leq \overline{x}$. Thus, $\mathbb{KR}$ is obtained by adjoining *improper* intervals $[\underline{x}, \overline{x}]$, $\underline{x} > \overline{x}$, to the set $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R},\ \underline{x} \leq \overline{x}\}$ of *proper* intervals and real numbers, the latter being identified with the corresponding degenerate intervals. We will denote elements of complete interval arithmetic as well as other objects formed of these elements by boldface letters, like the common intervals.

DEFINITION 5.1. An interval $\mathbf{x} \in \mathbb{KR}$ will be called *balanced*, if $\underline{\mathbf{x}} = -\overline{\mathbf{x}}$.*

DEFINITION 5.2. The absolute value (module) of an interval $\mathbf{x} \in \mathbb{KR}$ is $|\mathbf{x}| := \max\{|\underline{\mathbf{x}}|, |\overline{\mathbf{x}}|\}$.

DEFINITION 5.3. A mapping $\mathrm{dual} : \mathbb{KR} \to \mathbb{KR}$ that acts as

$$\mathrm{dual}\ \mathbf{x} := [\overline{\mathbf{x}}, \underline{\mathbf{x}}],$$

reversing the endpoints of the intervals, is called *dualization*.

The proper and improper intervals, the two "halves" of $\mathbb{KR}$, change places as the result of the dualization.

As in classical interval arithmetic, the inclusion order on $\mathbb{KR}$ is defined as follows:

$$\mathbf{x} \subseteq \mathbf{y} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \underline{\mathbf{x}} \geq \underline{\mathbf{y}} \quad \text{and} \quad \overline{\mathbf{x}} \leq \overline{\mathbf{y}}, \tag{5.4}$$

* Proper balanced intervals are usually called "symmetric," which is really inconsistent with the terminology in the other branches of the mathematical science. For instance, what is a *symmetric interval matrix* then? A matrix all whose entries being symmetric, in the above sense, intervals, or, alternatively, a matrix the values of whose entries being situated symmetrically with respect to the main diagonal? The latter is traditional understanding of the classical linear algebra, which is unlikely to be changed.

but complete interval arithmetic $\mathbb{KR}$ is a distributive conditionally complete lattice [12] with respect to this inclusion order, in contrast with classical interval arithmetic. In other words,

$$\bigwedge_{\gamma\in\Gamma} \mathbf{x}_\gamma := \inf_{\subseteq}\{\mathbf{x}_\gamma \mid \gamma\in\Gamma\} = \left[\inf_{\leq}\{\underline{\mathbf{x}}_\gamma \mid \gamma\in\Gamma\},\ \sup_{\leq}\{\overline{\mathbf{x}}_\gamma \mid \gamma\in\Gamma\}\right],$$

$$\bigvee_{\gamma\in\Gamma} \mathbf{x}_\gamma := \sup_{\subseteq}\{\mathbf{x}_\gamma \mid \gamma\in\Gamma\} = \left[\sup_{\leq}\{\underline{\mathbf{x}}_\gamma \mid \gamma\in\Gamma\},\ \inf_{\leq}\{\overline{\mathbf{x}}_\gamma \mid \gamma\in\Gamma\}\right]$$

are always elements from $\mathbb{KR}$ now, provided that $\{\mathbf{x}_\gamma \mid \gamma \in \text{index set } \Gamma\}$ is a bounded family of intervals from $\mathbb{KR}$.

Addition and multiplication by real numbers are defined on $\mathbb{KR}$ by

$$\mathbf{x}+\mathbf{y} := [\underline{\mathbf{x}}+\underline{\mathbf{y}},\ \overline{\mathbf{x}}+\overline{\mathbf{y}}], \tag{5.5}$$

$$\mu\cdot\mathbf{x} := \begin{cases} [\mu\,\underline{\mathbf{x}},\ \mu\,\overline{\mathbf{x}}], & \text{if } \mu\geq 0,\\ [\mu\,\overline{\mathbf{x}},\ \mu\,\underline{\mathbf{x}}], & \text{otherwise.}\end{cases}$$

Thus, every element $\mathbf{x}$ from $\mathbb{KR}$ has a unique algebraic opposite element, denoted "opp $\mathbf{x}$," and

$$\mathbf{x}+\operatorname{opp}\mathbf{x}=0 \quad\Rightarrow\quad \operatorname{opp}[\underline{x},\overline{x}] = [-\underline{x},-\overline{x}].$$

For example, opp $[1,2] = [-1,-2]$, i.e. an improper "backward" interval. There holds

$$\begin{aligned}\operatorname{opp}(\operatorname{opp}\mathbf{x}) &= \mathbf{x},\\ \operatorname{opp}(\lambda\mathbf{x}) &= \lambda\operatorname{opp}\mathbf{x}, \qquad \text{for } \lambda\in\mathbb{R},\\ \operatorname{dual}(\mathbf{x}+\mathbf{y}) &= \operatorname{dual}\mathbf{x}+\operatorname{dual}\mathbf{y},\\ \operatorname{opp}(\mathbf{x}+\mathbf{y}) &= \operatorname{opp}\mathbf{x}+\operatorname{opp}\mathbf{y}.\end{aligned}$$

Also,

$$\begin{aligned}\mathbf{x}\subseteq\mathbf{y}\ \Rightarrow\quad -\mathbf{x} &\subseteq -\mathbf{y},\\ \operatorname{dual}\mathbf{x} &\supseteq \operatorname{dual}\mathbf{y},\\ \operatorname{opp}\mathbf{x} &\supseteq \operatorname{opp}\mathbf{y},\\ \lambda\mathbf{x} &\subseteq \lambda\mathbf{y}, \qquad \text{for } \lambda\in\mathbb{R}.\end{aligned}$$

Sometimes, we denote for brevity the inverse operation for addition, i.e. the inner (algebraic) difference in $\mathbb{KR}$, by $\ominus$, so that

$$\mathbf{x}\ominus\mathbf{y} := \mathbf{x}+\operatorname{opp}\mathbf{y}.$$

The addition (5.5) is obviously commutative and associative. Besides, with respect to addition, complete interval arithmetic $\mathbb{KR}$ becomes a commutative group,

Table 1. Multiplication in complete interval arithmetic.

$\cdot$	$\mathbf{y} \in \mathcal{P}$	$\mathbf{y} \in \mathcal{Z}$	$\mathbf{y} \in -\mathcal{P}$	$\mathbf{y} \in$ dual $\mathcal{Z}$
$\mathbf{x} \in \mathcal{P}$	$[\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}]$
$\mathbf{x} \in \mathcal{Z}$	$[\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\min\{\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\}, \max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\}]$	$[\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}]$	0
$\mathbf{x} \in -\mathcal{P}$	$[\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}]$	$[\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}]$
$\mathbf{x} \in$ dual $\mathcal{Z}$	$[\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}]$	0	$[\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\}, \min\{\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\}]$

which is isomorphic to the additive group of the standard linear space $\mathbb{R}^2$. The consequence of this is, among others, the usual possibility to rearrange terms from one side to the other side "with the opposite sign" in an interval equation (inequality, inclusion) for which we seek a formal solution in the sense of Definition 4.2.

The following lattice operation distributivity will be useful for us, too:

$$\mathbf{x} + (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} + \mathbf{y}) \vee (\mathbf{x} + \mathbf{z}), \tag{5.6}$$

$$\mathbf{x} + (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} + \mathbf{y}) \wedge (\mathbf{x} + \mathbf{z}). \tag{5.7}$$

Interval multiplication in $\mathbb{KR}$ is defined in a more sophisticated way. Sometimes, the minimax representation (5.22) of Section 5.4 is taken as such a definition. As an alternative, it makes sense to have a purely algebraic definition through the so-called *Cayley table*, i.e. a square table listing all the possible combinations of operands on inputs and the corresponding results inside the table. Let us separate in $\mathbb{KR}$ the following subsets:

$$\begin{aligned}
\mathcal{P} &:= \{\mathbf{x} \in \mathbb{KR} \mid (\underline{\mathbf{x}} \geq 0)\ \&\ (\overline{\mathbf{x}} \geq 0)\}, && \text{—positive intervals,}\\
\mathcal{Z} &:= \{\mathbf{x} \in \mathbb{KR} \mid \underline{\mathbf{x}} \leq 0 \leq \overline{\mathbf{x}}\}, && \text{—zero-containing intervals,}\\
-\mathcal{P} &:= \{\mathbf{x} \in \mathbb{KR} \mid -\mathbf{x} \in \mathcal{P}\}, && \text{—negative intervals,}\\
\text{dual } \mathcal{Z} &:= \{\mathbf{x} \in \mathbb{KR} \mid \text{dual } \mathbf{x} \in \mathcal{Z}\}, && \text{—intervals contained in the zero.}
\end{aligned}$$

Overall $\mathbb{KR} = \mathcal{P} \cup (-\mathcal{P}) \cup \mathcal{Z} \cup$ dual $\mathcal{Z}$. Then the multiplication in complete arithmetic is described by Table 1 [43].

As one can see, multiplication in the arithmetic $\mathbb{KR}$ allows nontrivial zero divisors, for instance, $[-1, 2] \cdot [5, -3] = 0$. Additionally, multiplication is both commutative and associative, like in $\mathbb{IR}$, but not every element $\mathbf{x}$ of $\mathbb{KR}$ has its inverse $\mathbf{x}^{-1}$. That is, the group with respect to operation "$\cdot$" is formed in $\mathbb{KR}$ only by intervals $\mathbf{x}$ with $\underline{\mathbf{x}}\,\overline{\mathbf{x}} > 0$ [42]. For example, $[1, 2]^{-1} = [1, \frac{1}{2}]$, while $[-2, 3]^{-1}$

does not exist. We denote for brevity the inverse operation for multiplication, i.e. the inner (algebraic) division in $\mathbb{KR}$, by $\oslash$, so that

$$\mathbf{x} \oslash \mathbf{y} \;:=\; \mathbf{x} \cdot \mathbf{y}^{-1}.$$

There also holds

$$\begin{aligned} \operatorname{dual} \mathbf{x} \cdot \operatorname{dual} \mathbf{y} &= \operatorname{dual} (\mathbf{xy}), \\ \operatorname{opp} \mathbf{x} \cdot \operatorname{opp} \mathbf{y} &= \operatorname{dual} (\mathbf{xy}), \\ \lambda\,(\mathbf{xy}) &= (\lambda \mathbf{x})\,\mathbf{y} \;=\; \mathbf{x}\,(\lambda \mathbf{y}), \qquad \text{for } \lambda \in \mathbb{R}. \end{aligned}$$

Table 1 may turn out rather awkward to implement and study the multiplication in the complete interval arithmetic. A. Lakeyev in [57] proposed simple global formulas for the interval product in $\mathbb{KR}$, which can be more suitable in a number of situations. We remind the following definition [11]:

DEFINITION 5.4. For a real number x, the values

$$\begin{aligned} x^{+} &:= \max\{x, 0\}, \\ x^{-} &:= \max\{-x, 0\} \end{aligned}$$

are called *positive* and *negative* parts of x respectively.

PROPOSITION 5.1 [57]. *For any intervals* $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$, *the following representation holds*

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} \;=\; & [\max\{\underline{\mathbf{x}}^{+}\underline{\mathbf{y}}^{+}, \overline{\mathbf{x}}^{-}\overline{\mathbf{y}}^{-}\} - \max\{\overline{\mathbf{x}}^{+}\underline{\mathbf{y}}^{-}, \underline{\mathbf{x}}^{-}\overline{\mathbf{y}}^{+}\}, \\ & \max\{\overline{\mathbf{x}}^{+}\overline{\mathbf{y}}^{+}, \underline{\mathbf{x}}^{-}\underline{\mathbf{y}}^{-}\} - \max\{\underline{\mathbf{x}}^{+}\overline{\mathbf{y}}^{-}, \overline{\mathbf{x}}^{-}\underline{\mathbf{y}}^{+}\}]. \end{aligned}$$

If one of the intervals $\mathbf{x}$, $\mathbf{y}$ *is proper, then*

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} \;=\; & [\,\underline{\mathbf{x}}^{+}\underline{\mathbf{y}}^{+} + \overline{\mathbf{x}}^{-}\overline{\mathbf{y}}^{-} - \max\{\overline{\mathbf{x}}^{+}\underline{\mathbf{y}}^{-}, \underline{\mathbf{x}}^{-}\overline{\mathbf{y}}^{+}\}, \\ & \max\{\overline{\mathbf{x}}^{+}\overline{\mathbf{y}}^{+}, \underline{\mathbf{x}}^{-}\underline{\mathbf{y}}^{-}\} - \underline{\mathbf{x}}^{+}\overline{\mathbf{y}}^{-} - \overline{\mathbf{x}}^{-}\underline{\mathbf{y}}^{+}]. \end{aligned} \tag{5.8}$$

This formula is not simplified even if we know that both intervals $\mathbf{x}$, $\mathbf{y}$ *are proper. If one of the intervals* $\mathbf{x}$, $\mathbf{y}$ *is proper while the other is improper, then*

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} \;=\; & [\underline{\mathbf{x}}^{+}\underline{\mathbf{y}}^{+} + \overline{\mathbf{x}}^{-}\overline{\mathbf{y}}^{-} - \overline{\mathbf{x}}^{+}\underline{\mathbf{y}}^{-} - \underline{\mathbf{x}}^{-}\overline{\mathbf{y}}^{+}, \\ & \overline{\mathbf{x}}^{+}\overline{\mathbf{y}}^{+} + \underline{\mathbf{x}}^{-}\underline{\mathbf{y}}^{-} - \underline{\mathbf{x}}^{+}\overline{\mathbf{y}}^{-} - \overline{\mathbf{x}}^{-}\underline{\mathbf{y}}^{+}]. \end{aligned} \tag{5.9}$$

The advantage of Lakeyev formulas is their global character. They give us single unified expressions for the interval product $\mathbf{x} \cdot \mathbf{y}$ over all of the domain of both $\mathbf{x}$ and $\mathbf{y}$, while the representation through Table 1 has piecewise character. The latter is quite embarrassing e.g. when studying differentiability properties, smoothness and their analogues, when computing the derivatives, etc.

Definitions of the interval subtraction and division in $\mathbb{KR}$ are the same as in classical interval arithmetic $\mathbb{IR}$:

$$\mathbf{x} - \mathbf{y} := \mathbf{x} + (-1) \cdot \mathbf{y},$$
$$\mathbf{x} / \mathbf{y} := \mathbf{x} \cdot [1 / \overline{\mathbf{y}}, 1 / \underline{\mathbf{y}}] \qquad \text{for } \underline{\mathbf{y}}\,\overline{\mathbf{y}} > 0.$$

Clearly, in $\mathbb{KR}$ one should sharply distinguish between the interval operations "$\oslash$" and "/" as well as between "$\ominus$" and "$-$".

The property of fundamental importance is that the inclusion monotonicity holds in complete interval arithmetic too:

$$\mathbf{x} \subseteq \mathbf{x}', \quad \mathbf{y} \subseteq \mathbf{y}' \quad \Rightarrow \quad \mathbf{x} \star \mathbf{y} \subseteq \mathbf{x}' \star \mathbf{y}'$$

for $\star \in \{+, -, \cdot, /\}$ and any $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbb{KR}$.

Like in classical interval arithmetic, addition and multiplication in complete interval arithmetic are not connected by the distributive law in general. Instead we have weaker subdistributivity and superdistributivity properties [25], [43]:

$$\text{if } \mathbf{x} \text{ is proper, then} \quad \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} \tag{5.10}$$

—subdistributivity,

$$\text{if } \mathbf{x} \text{ is improper, then} \quad \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \supseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} \tag{5.11}$$

—superdistributivity.

These inclusions turn to the equalities, in particular, if the interval $\mathbf{x}$ degenerate into a point, i.e. $\mathbf{x} = x \in \mathbb{R}$:

$$x \cdot (\mathbf{y} + \mathbf{z}) = x \cdot \mathbf{y} + x \cdot \mathbf{z}. \tag{5.12}$$

Unfortunately, the distributivity of multiplication with respect to the lattice operations $\vee$ and $\wedge$ is not the case in general. For example,

$$[-1, 1] \cdot \big(1 \wedge (-1)\big) = 0 \neq [-1, 1] = ([-1, 1] \cdot 1) \wedge \big([-1, 1] \cdot (-1)\big).$$

Nonetheless, if $\mathbf{x}$ is either positive or negative interval (i.e., from $\mathcal{P} \cup (-\mathcal{P})$), then

$$\mathbf{x} \cdot (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \vee (\mathbf{x} \cdot \mathbf{z}), \tag{5.13}$$
$$\mathbf{x} \cdot (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \wedge (\mathbf{x} \cdot \mathbf{z}). \tag{5.14}$$

At the same time, if $\mathbf{x}$ is proper, then (5.13) is still valid, while

$$\mathbf{x} \cdot (\mathbf{y} \wedge \mathbf{z}) \subseteq (\mathbf{x} \cdot \mathbf{y}) \wedge (\mathbf{x} \cdot \mathbf{z}),$$

and if $\mathbf{x}$ is improper, then (5.14) is still valid, while

$$\mathbf{x} \cdot (\mathbf{y} \vee \mathbf{z}) \supseteq (\mathbf{x} \cdot \mathbf{y}) \vee (\mathbf{x} \cdot \mathbf{z}).$$

Table 2. Composition of involutions of $\mathbb{KR}$.

$\circ$	id	–	opp	dual
id	id	–	opp	dual
–	–	id	dual	opp
opp	opp	dual	id	–
dual	dual	opp	–	id

For detailed proofs see, e.g., [28].

Finally, it is quite instructive to note that the set of basic involutions of the complete interval arithmetic $\mathbb{KR}$—negation "–", opposition "opp", dualization "dual" as well as identity "id"—multiply (compose) according to the Cayley table (see Table 2).

In other words, the multiplicative structure on their set coincides with the well-known Klein's "four-group."

5.3. Interval Vectors and Matrices

Arithmetical operations with vectors and matrices made up of the elements of $\mathbb{KR}$ are defined similar to those in classical interval arithmetic (see, e.g., [2], [68], [69]).

DEFINITION 5.5. The sum (difference) of the two interval matrices of identical size is an interval matrix of the same size formed by the elementwise sums (differences) of the operands. The product of interval matrices $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} = (\mathbf{x}_{ij}) \in \mathbb{KR}^{m\times l}$, $\mathbf{Y} = (\mathbf{y}_{ij}) \in \mathbb{KR}^{l\times n}$, is a matrix $\mathbf{X}\cdot\mathbf{Y} = \mathbf{Z} = (\mathbf{z}_{ij}) \in \mathbb{KR}^{m\times n}$, such that

$$\mathbf{z}_{ij} = \sum_{k=1}^{l} \mathbf{x}_{ik}\mathbf{y}_{kj}. \tag{5.15}$$

The well-known feature of the interval matrix multiplication is the absence of associativity. This is also valid for the complete interval arithmetic, although for some important particular cases the associativity still takes place. In particular, there holds

PROPOSITION 5.2. *If* $X \in \mathbb{R}^{m\times l}$, $\mathbf{Y} \in \mathbb{KR}^{l\times k}$, $Z \in \mathbb{R}^{k\times n}$, *then*

$$(X\mathbf{Y})Z = X(\mathbf{Y}Z).$$

Proof.

$$\big((X\mathbf{Y})Z\big)_{ij} = \sum_{\nu} (X\mathbf{Y})_{i\nu} Z_{\nu j} = \sum_{\nu} \left(\sum_{\mu} X_{i\mu}\mathbf{Y}_{\mu\nu}\right) Z_{\nu j}$$

$$
\begin{aligned}
&= \sum_{\nu}\sum_{\mu}(X_{i\mu}\mathbf{Y}_{\mu\nu})Z_{\nu j} \;=\; \sum_{\nu}\sum_{\mu}X_{i\mu}(\mathbf{Y}_{\mu\nu}Z_{\nu j})\\
&= \sum_{\mu}\sum_{\nu}X_{i\mu}(\mathbf{Y}_{\mu\nu}Z_{\nu j}) \overset{(*)}{=} \sum_{\mu}X_{i\mu}\sum_{\nu}(\mathbf{Y}_{\mu\nu}Z_{\nu j})\\
&= \sum_{\mu}X_{i\mu}(\mathbf{YZ})_{\mu j} \;=\; \big(X(\mathbf{YZ})\big)_{ij},
\end{aligned}
$$

where to carry out the common multiplier $X_{i\mu}$ out of the sum in the equality $(*)$ we make use of the distributivity relation (5.12). □

Additionally, we also need the operations of taking the midpoint and radius of an interval:

$$\operatorname{mid}\mathbf{x} := \tfrac{1}{2}(\overline{\mathbf{x}}+\underline{\mathbf{x}}),$$

$$\operatorname{rad}\mathbf{x} := \tfrac{1}{2}(\overline{\mathbf{x}}-\underline{\mathbf{x}}).$$

As usual, with respect to the interval vectors and matrices these operations, as well as "dual", "pro", "opp", will be applied componentwise and elementwise.

Inclusion ordering on the sets of interval vectors and matrices with the elements from $\mathbb{KR}$ are, by definition, the direct products [12] of the one-dimensional inclusion orders on the separate components $\mathbb{KR}$, so that

$$\mathbf{x}\subseteq\mathbf{y} \quad\Longleftrightarrow\quad \mathbf{x}_i\subseteq\mathbf{y}_i \quad \text{for all } i.$$

We shall therefore understand the operations $\vee$ and $\wedge$ applied to interval vectors in the componentwise manner, i.e.

$$\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_n\end{pmatrix}\vee\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\\ \mathbf{y}_n\end{pmatrix}=\begin{pmatrix}\mathbf{x}_1\vee\mathbf{y}_1\\ \mathbf{x}_2\vee\mathbf{y}_2\\ \vdots\\ \mathbf{x}_n\vee\mathbf{y}_n\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_n\end{pmatrix}\wedge\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\\ \mathbf{y}_n\end{pmatrix}=\begin{pmatrix}\mathbf{x}_1\wedge\mathbf{y}_1\\ \mathbf{x}_2\wedge\mathbf{y}_2\\ \vdots\\ \mathbf{x}_n\wedge\mathbf{y}_n\end{pmatrix}.$$

Also, the binary relation "$\leq$" as well as the action of the operations "dual", "pro", "mid", and "rad" on interval vectors and matrices will be taken componentwise.

The distance—dist $(\cdot\,,\cdot)$—between the elements of the complete interval arithmetic $\mathbb{KR}$ is known to be introduced as follows [43]:

$$\operatorname{dist}(\mathbf{x},\mathbf{y}) := \max\{|\underline{\mathbf{x}}-\underline{\mathbf{y}}|,\ |\overline{\mathbf{x}}-\overline{\mathbf{y}}|\} = |\mathbf{x}\ominus\mathbf{y}|.$$

With such metric, for any intervals $\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}'\in\mathbb{KR}$ the following inequalities are valid [43]

$$\operatorname{dist}(\mathbf{xy},\mathbf{xy}') \;\leq\; |\mathbf{x}|\cdot\operatorname{dist}(\mathbf{y},\mathbf{y}'), \tag{5.16}$$

$$\operatorname{dist}(\mathbf{x}+\mathbf{y},\ \mathbf{x}'+\mathbf{y}') \;\leq\; \operatorname{dist}(\mathbf{x},\mathbf{x}')+\operatorname{dist}(\mathbf{y},\mathbf{y}'). \tag{5.17}$$

As for the topology on the multidimensional interval space $\mathbb{KR}^n$, it can be defined in two (basically equivalent) ways. The standard way is to introduce the common metric*

$$\operatorname{dist}(\mathbf{x},\mathbf{y}) := \max\{\|\underline{\mathbf{x}} - \underline{\mathbf{y}}\|,\ \|\overline{\mathbf{x}} - \overline{\mathbf{y}}\|\}, \qquad \mathbf{x},\mathbf{y} \in \mathbb{KR}^n, \tag{5.18}$$

where $\|\cdot\|$ is an absolute vector norm on $\mathbb{R}^n$. Sometimes, however, it proves helpful and convenient to work with a vector-valued distance,—*pseudometric* according to the terminology by L. Collatz [16]**,—which is defined on $\mathbb{KR}^n$ as

$$\operatorname{Dist}(\mathbf{x},\mathbf{y}) := \begin{pmatrix} \operatorname{dist}(\mathbf{x}_1,\mathbf{y}_1) \\ \vdots \\ \operatorname{dist}(\mathbf{x}_n,\mathbf{y}_n) \end{pmatrix} \in \mathbb{R}^n. \tag{5.19}$$

All the interval arithmetic operations, matrix-vector operations on $\mathbb{KR}^n$ as well as the operations "$\vee$", "$\wedge$", "dual", "pro", and "opp" turns out to be continuous both with respect to the metric (5.18) (see, in particular, [43]) and pseudometric (5.19).

Notice that the estimate (5.16) can be carried over to the multidimensional case as well, if the distance between $\mathbf{x},\mathbf{y} \in \mathbb{KR}^n$ is understood as the pseudometric $\operatorname{Dist}(\cdot,\cdot)$ and the module of a matrix is taken elementwise. More precisely, there holds

PROPOSITION 5.3. *For any interval matrix* $\mathbf{P} = (\mathbf{p}_{ij}) \in \mathbb{KR}^{n\times n}$ *and any interval vectors* $\mathbf{x},\mathbf{y} \in \mathbb{KR}^n$, *we have*

$$\operatorname{Dist}(\mathbf{Px},\mathbf{Py}) \leq |\mathbf{P}| \cdot \operatorname{Dist}(\mathbf{x},\mathbf{y}). \tag{5.20}$$

Proof. Indeed, in view of the inequalities (5.16)–(5.17), we can conclude that

$$\begin{aligned} \operatorname{dist}\left((\mathbf{Px})_i,(\mathbf{Py})_i\right) &= \operatorname{dist}\left(\sum_{j=1}^n \mathbf{p}_{ij}\mathbf{x}_j,\ \sum_{j=1}^n \mathbf{p}_{ij}\mathbf{y}_j\right) \\ &\leq \sum_{j=1}^n \operatorname{dist}(\mathbf{p}_{ij}\mathbf{x}_j,\ \mathbf{p}_{ij}\mathbf{y}_j) \\ &\leq \sum_{j=1}^n |\mathbf{p}_{ij}| \cdot \operatorname{dist}(\mathbf{x}_j,\mathbf{y}_j) \end{aligned}$$

for all $i \in \{1,2,\ldots,n\}$. That proves‡ the multidimensional estimate (5.20). □

* For the space $\mathbb{IR}^n$ this metric coincides with Hausdorff distance between the interval vectors as axis-aligned boxes in $\mathbb{R}^n$.

** Pseudometric spaces are often called *multimetric spaces* or *hypermetric spaces* in the modern literature on functional analysis.

‡ For the case of classical interval arithmetic $\mathbb{IR}$, the inequality (5.20) is well-known, but for the complete interval arithmetic $\mathbb{KR}$ it has been neither mentioned nor used by anybody in the multidimensional case.

5.4. KAUCHER COMPLETE ARITHMETIC IS MINIMAX INTERVAL ARITHMETIC

With the use of lattice maximum operation (5.3), we can rewrite the fundamental property (5.1) defining classical interval arithmetic in the form:

$$\mathbf{x} \star \mathbf{y} = \bigvee_{x \in \mathbf{x}} \bigvee_{y \in \mathbf{y}} (x \star y). \tag{5.21}$$

The above facilitates recognizing the most wonderful fact with complete interval arithmetic, which generalizes the formulas (5.1) and (5.21). Namely, for any $\star \in \{+, -, \cdot, /\}$, the following relationship holds:

$$\mathbf{x} \star \mathbf{y} = \bigwedge\!\!\!\bigvee^{\mathbf{x}}_{x \in \operatorname{pro} \mathbf{x}} \bigwedge\!\!\!\bigvee^{\mathbf{y}}_{y \in \operatorname{pro} \mathbf{y}} (x \star y), \tag{5.22}$$

where

$$\bigwedge\!\!\!\bigvee^{\mathbf{x}} := \begin{cases} \bigvee, & \text{if } \mathbf{x} \text{ is proper,} \\ \bigwedge, & \text{otherwise,} \end{cases} \quad \text{—conditional extremum operation,}$$

$$\operatorname{pro} \mathbf{x} := \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \text{ is proper,} \\ \operatorname{dual} \mathbf{x}, & \text{otherwise,} \end{cases} \quad \text{—proper projection of the interval.}$$

Discovered first by E. Kaucher [44], this representation expresses the connection between the interval operation $\mathbf{x} \star \mathbf{y}$ and the results of the point operations $x \star y$ for $x \in \operatorname{pro} \mathbf{x}$ and $y \in \operatorname{pro} \mathbf{y}$. Notice that, as it follows from (5.22), complete interval arithmetic is the desired minimax interval arithmetic! Indeed, in $\mathbb{KR}$ *endpoints of a resulting interval are minimax and maximin of the results of the point arithmetical operation, if of the intervals under operation one is proper and the other is improper*. For example,

$$[-3, 5] \cdot [2, -1] = 0 = \left[\min_{x \in [-3,5]} \max_{y \in [-1,2]} x \cdot y, \ \max_{x \in [-3,5]} \min_{y \in [-1,2]} x \cdot y \right],$$

in accordance with the multiplication table.*

We now lead towards a natural interpretation of Kaucher complete interval arithmetic. Namely, the improper intervals of $\mathbb{KR}$ are not something exotic at all. Taken from set-theoretical standpoint, they may be considered as usual sets of points on $\mathbb{R}$ bounded by two real numbers, while their "backward" direction is only an indication that they act in arithmetic operations, etc., in a special ("minimax") way, not like the classical proper intervals of $\mathbb{IR}$.

Is it possible to make use of the above properties in the computation of minimaxes for more complex composite expressions? The answer is positive on the whole, but it is not so simple and straightforward as for the classical interval arithmetic and "pure" minima and maxima of the functions. The corresponding

* By the way, we formed the sign of our conditional extremum operation by coupling the symbols "$\vee$" and "$\wedge$" together.

(very subtle and sophisticated) theory has been elaborated in the works of Spanish researchers headed by E. Gardeñes and M. Sainz, and its most complete and correct exposition can be found in the recent publications [24], [122]. Although for the elementary arithmetic operations—addition, multiplication, subtraction and division—minimum and maximum commute with each others, they are well known to be non-commutable for the general case of complex rational expressions. Even imposing the stringent requirement that every variable occurs only once does not redeem the position. The following cogent example is borrowed from the survey [122].

We consider a function of four variables

$$\phi(x_1, x_2, x_3, x_4) = (x_1 + x_2)(x_3 + x_4).$$

If the variables have the domains $x_1 \in [-2, 2]$, $x_2 \in [-1, 1]$, $x_3 \in [-1, 1]$, $x_4 \in [-2, 2]$, then

$$\bigvee_{x_1 \in [-2,2],\ x_3 \in [-1,1]} \ \bigwedge_{x_2 \in [-1,1],\ x_4 \in [-2,2]} \phi(x_1, x_2, x_3, x_4) = \left[\frac{3}{2}, -\frac{3}{2}\right],$$

$$\bigwedge_{x_2 \in [-1,1],\ x_4 \in [-2,2]} \ \bigvee_{x_1 \in [-2,2],\ x_3 \in [-1,1]} \phi(x_1, x_2, x_3, x_4) = \left[-\frac{3}{2}, \frac{3}{2}\right],$$

which differs from the result of the corresponding "natural interval extension" for the expression under study:

$$\phi([-2, 2], [1, -1], [-1, 1], [2, -2]) = 0.$$

Still, using induction over syntactic tree* of the expression, it is not hard to derive from (5.22) that, if a rational expression $f(x, y) = f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ has at most one occurrence of each of the variables x_i, y_j and to the first power only, then

$$\bigvee_{x \in \mathbf{x}} \bigwedge_{y \in \mathbf{y}} f(x, y) \subseteq f(\mathbf{x}, \operatorname{dual} \mathbf{y}) \subseteq \bigwedge_{y \in \mathbf{y}} \bigvee_{x \in \mathbf{x}} f(x, y), \tag{5.23}$$

for any proper interval vectors $\mathbf{x} \in \mathbb{IR}^p$, $\mathbf{y} \in \mathbb{IR}^q$, i.e., in the extended form,

$$\left[\min_{x \in \mathbf{x}} \max_{y \in \mathbf{y}} f(x, y),\ \max_{x \in \mathbf{x}} \min_{y \in \mathbf{y}} f(x, y)\right] \subseteq f(\mathbf{x}, \operatorname{dual} \mathbf{y}),$$

$$f(\mathbf{x}, \operatorname{dual} \mathbf{y}) \subseteq \left[\max_{x \in \mathbf{x}} \min_{y \in \mathbf{y}} f(x, y),\ \min_{x \in \mathbf{x}} \max_{y \in \mathbf{y}} f(x, y)\right].$$

The more complex cases which can also be proved by induction. Given a rational expression $f(x, y) = f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ that has only one occurrence of each of the variables y_i (if at all) and to the first power only, for any proper interval vectors $\mathbf{x} \in \mathbb{IR}^p$, $\mathbf{y} \in \mathbb{IR}^q$, we have

$$\left[\min_{x \in \mathbf{x}} \max_{y \in \mathbf{y}} f(x, y),\ \max_{x \in \mathbf{x}} \min_{y \in \mathbf{y}} f(x, y)\right] \subseteq f(\mathbf{x}, \operatorname{dual} \mathbf{y}). \tag{5.24}$$

* Also called Kantorovich tree, see [7].

Given a rational expression $f(x,y) = f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ that has only one occurrence of each of the variables x_i (if at all) and to the first power only, for any proper interval vectors $\mathbf{x} \in \mathbb{IR}^p$, $\mathbf{y} \in \mathbb{IR}^q$, we have

$$\left[\min_{x\in\mathbf{x}} \max_{y\in\mathbf{y}} f(x,y),\ \max_{x\in\mathbf{x}} \min_{y\in\mathbf{y}} f(x,y)\right] \supseteq f(\mathbf{x}, \operatorname{dual} \mathbf{y}). \tag{5.25}$$

The relations (5.24) and (5.25) have been first obtained by E. Gardeñes, M. Sainz and their co-workers (in the terms which are different from ours though), and their detailed proofs can be found [26], [28], [122].

The history of the problem is quite amusing. In the early publications of the Spanish researchers [25], [26], it was mistakenly stated that

$$\bigvee_{x\in\mathbf{x}} \bigwedge_{y\in\mathbf{y}} f(x,y) = f(\mathbf{x}, \operatorname{dual} \mathbf{y}) = \bigwedge_{y\in\mathbf{y}} \bigvee_{x\in\mathbf{x}} f(x,y),$$

if every variable x_i, y_i occurs in f no more than once and to the first power. A new, previously unknown, minimax theorem was thereby claimed! The error has been fixed only a decade later, while the paper with the correct formulation is dated by 1999 [122]. No wonder, some people managed to make use of the wrong result in their works.

To summarize the section, we may conclude that Kaucher complete interval arithmetic, although not correcting all the drawbacks of classical interval arithmetic, is still much more suitable and fitting for

1) the computation of formal solutions to interval systems of equations,
2) the solution of the minimax problems.

5.5. Characteristic Matrix and Right-Hand Side Vector

As an immediate demonstration of the advantages Kaucher complete interval arithmetic grants us, we introduce

DEFINITION 5.6 [110], [117], [118]. The interval matrix $\mathbf{A}^{\mathbf{c}}$ and interval vector $\mathbf{b}^{\mathbf{c}}$ defined as

$$\mathbf{A}^{\mathbf{c}} := \mathbf{A}^{\forall} + \operatorname{dual} \mathbf{A}^{\exists}, \qquad \mathbf{b}^{\mathbf{c}} := \operatorname{dual} \mathbf{b}^{\forall} + \mathbf{b}^{\exists},$$

will be called *characteristic* for the AE-solution set of the interval linear system (3.14) determined by the disjoint decompositions of $\mathbf{A}$ to $\mathbf{A}^{\forall}$ and $\mathbf{A}^{\exists}$, of $\mathbf{b}$ to $\mathbf{b}^{\forall}$ and $\mathbf{b}^{\exists}$.

The concept we have specified by Definition 5.6 is so significant in our further theory that it makes sense to discuss it more thoroughly. We emphasize that pointing out the characteristic interval matrix and right-hand side vector completely determines an AE-solution set to the interval linear system of equations along with the triple described in Section 3, i.e. with the quantifier matrix α and vector β, partitions

of the index sets of the interval parameter matrix and right-hand side vector as well as their disjoint decompositions. Setting the characteristic matrix and vector gives even more information by jointly indicating both the type of the interval uncertainty and the values of these intervals. Therefore, it is quite correct to speak of an AE-solution set to some interval system of equation that *correspond to a characteristic matrix and a right-hand side vector of the interval parameters* and write $\Xi(\mathbf{A}^{\mathfrak{c}}, \mathbf{b}^{\mathfrak{c}})$ without explicit writing out the interval system and distribution of the uncertainty types in it. The new concepts and terminology proves to be extremely beneficial when applied to the interval linear systems of equations as will be showed, e.g., in Section 7.4.

To illustrate the foregoing, we give a concise form of the main characterization (3.23):

THEOREM 5.1. *The point* $x \in \mathbb{R}^n$ *belongs to the solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *if and only if*

$$\mathbf{A}^{\mathfrak{c}} \cdot x \subseteq \mathbf{b}^{\mathfrak{c}} \tag{5.26}$$

in Kaucher complete interval arithmetic.

Proof. Notice that

$$\operatorname{opp}(-\mathbf{v}) = \operatorname{dual} \mathbf{v}$$

for any interval $\mathbf{v} \in \mathbb{KR}$. Therefore, adding $(\operatorname{dual} \mathbf{b}^{\forall} + \operatorname{dual} (\mathbf{A}^{\exists} \cdot x))$ to both sides of the inclusion (3.23) results in the following equivalent inclusion in the complete interval arithmetic

$$\mathbf{A}^{\forall} \cdot x + \operatorname{dual} (\mathbf{A}^{\exists} \cdot x) \subseteq \operatorname{dual} \mathbf{b}^{\forall} + \mathbf{b}^{\exists}. \tag{5.27}$$

But $\operatorname{dual} (\mathbf{A}^{\exists} \cdot x) = (\operatorname{dual} \mathbf{A}^{\exists}) \cdot x$, since x is a point vector. Instead of (5.27) we may thus write

$$\mathbf{A}^{\forall} \cdot x + (\operatorname{dual} \mathbf{A}^{\exists}) \cdot x \subseteq \operatorname{dual} \mathbf{b}^{\forall} + \mathbf{b}^{\exists}.$$

Finally, in the left-hand side we can avail ourselves of the distributivity with respect to the point variable x, arriving from (3.23) to the equivalent inclusion

$$(\mathbf{A}^{\forall} + \operatorname{dual} \mathbf{A}^{\exists}) \cdot x \subseteq \operatorname{dual} \mathbf{b}^{\forall} + \mathbf{b}^{\exists},$$

which coincides with (5.26). □

6. Inner Estimation of the Solution Sets

6.1. Formal Approach

In the approach developed in this section to the inner estimation problem (4.1), we change it for the problem of finding formal solutions to a special systems of equations in complete interval arithmetic $\mathbb{KR}$, thus reducing the original problem to a purely algebraic problem of numerical analysis. The cornerstone of this technique, which we will call *formal approach*, is the following

THEOREM 6.1 [104], [105], [107], [111], [112]. *Let* $\mathbf{A}^{\mathrm{c}}$ *and* $\mathbf{b}^{\mathrm{c}}$ *be the characteristic matrix and right-hand side vector of an interval linear system* $\mathbf{A}x = \mathbf{b}$ *which correspond to its AE-solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. *If a proper interval vector* $\mathbf{x}$ *is a formal solution to the equation*

$$\mathbf{A}^{\mathrm{c}} x = \mathbf{b}^{\mathrm{c}}, \tag{6.1}$$

then $\mathbf{x} \subseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, *i.e.* $\mathbf{x}$ *is an inner interval estimate of the solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

DEFINITION 6.1. For the interval system $\mathbf{A}x = \mathbf{b}$, we will call the equation (6.1) the *dualization equation* that corresponds to the AE-solution set of the type $\alpha\beta$ of the interval system $\mathbf{A}x = \mathbf{b}$, or, equivalently, to the distribution of the uncertainty types over the interval elements specified by the quantifiers α and β.

Proof. Let a proper interval vector $\mathbf{x}$ be a formal solution to the system (6.1) and $\tilde{x} \in \mathbf{x}$. Then, in view of inclusion monotonicity of interval arithmetical operations in $\mathbb{KR}$, we have

$$\mathbf{A}^{\mathrm{c}} \cdot \tilde{x} \subseteq \mathbf{A}^{\mathrm{c}} \cdot \mathbf{x} = \mathbf{b}^{\mathrm{c}},$$

that is, $\tilde{x} \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ by Theorem 5.1. □

It is worth listing the most significant particular cases of the above general result:

- If a proper interval vector $\mathbf{x}$ is a formal solution to the equation

  $$(\text{dual } \mathbf{A})\, x = \mathbf{b},$$

  then $\mathbf{x} \subseteq \Xi_{uni}(\mathbf{A}, \mathbf{b})$, i.e., $\mathbf{x}$ is an inner interval estimate of the united solution set to the equation $\mathbf{A}x = \mathbf{b}$.[⋆]
- If a proper interval vector $\mathbf{x}$ is a formal solution to the equation

  $$\mathbf{A}x = \mathbf{b},$$

  then $\mathbf{x} \subseteq \Xi_{tol}(\mathbf{A}, \mathbf{b})$, i.e., $\mathbf{x}$ is an inner interval estimate for the tolerable solution set to the equation $\mathbf{A}x = \mathbf{b}$ (or, in other words, a solution to the *tolerance problem* for the equation $\mathbf{A}x = \mathbf{b}$; see also [119]).[⋆⋆]

For example, one can simply check over, by direct substitution, that the formal solution to the model system (3.21) is the proper interval vector $\left(\left[-\frac{1}{3}, \frac{1}{3}\right], \left[-\frac{1}{3}, \frac{1}{3}\right]\right)^{\top}$. Figure 3 demonstrates that it really gives a good inner approximation for the tolerable solution set Ξ_{tol} of this system. On the other hand, if the

[⋆] This elegant and very practical result has been advanced simultaneously and independently by the author and by L. Kupriyanova, which was revealed at the international conference INTERVAL'94, St.-Petersburg, Russia, March 7-10, 1994 (published later in [53] and [106]).

[⋆⋆] V. Zyuzin [131] seems to be the first who pointed to the possibility of such an estimation for the tolerable solution set, although he had done that very briefly (in one sentence) and in an indirect form.

formal solution of the dualization equation (6.1) does not exist or is not entirely proper, it does not necessarily imply that the corresponding solution set is empty and the inner estimation problem (4.3) is incompatible.

The problem of computing formal solutions to interval linear systems in Kaucher complete interval arithmetic turned out to be NP-hard, which was established by A. Lakeyev in [57], [58]. Nonetheless, a number of efficient numerical methods has been constructed for computing the formal solutions in the last decade, and they work well enough provided the matrix of the input interval linear system is not "too wide". These are *subdifferential Newton method* [105], [106], [109] (which turns into quasidifferential Newton method in the general case) and various versions of the *stationary single-step iteration methods*, both of Jacobi type [52], [65], [96] and those based on splitting of the interval matrix of the equation [41], [105], [109], [113]. To sum up, we can assert that, for the square interval linear systems of the form

$$\mathbf{C}x = \mathbf{d}, \tag{6.2}$$

$\mathbf{C} \in \mathbb{KR}^{n\times n}$, $\mathbf{d} \in \mathbb{KR}^{n}$, the problem of computing the formal solutions can be solved (more or less) satisfactory.*

It is pertinent to note that, when solving the dualization equation in the general case, we could hardly use symbolic (computer algebra) manipulations, elimination methods, etc. The explanation is that algebraic properties of $\mathbb{KR}$ are still poor. Though they are better than those of classical interval arithmetic, the lack of distributivity makes it impossible even such simplest operation as the reduction of similar terms. This is the reason why all the above mentioned algorithms for computing formal solutions to interval linear systems are essentially *numerical*, while few attempts to develop symbolic methods for finding formal solutions (such as e.g. [81]) were of little success.

What about interval nonlinear systems of equations? We are able to produce immediate generalizations of Theorem 6.1 for important particular cases of the inner estimation of the united, tolerable and controllable solution sets (3.8)–(3.10) to interval nonlinear systems.

PROPOSITION 6.1 [99], [100]. *Let the mapping F be such that each of the variables $a_1, a_2, \ldots, a_l$ occurs only once (if at all) and to the first power in at most one of the component expressions $F_1, F_2, \ldots, F_m$. If a proper interval vector $\mathbf{x}$ is a formal solution to the equation*

$$F(\text{dual } \mathbf{a}, x) = \mathbf{b},$$

then $\mathbf{x} \subseteq \Xi_{uni}(F, \mathbf{a}, \mathbf{b})$, i.e., $\mathbf{x}$ is an inner interval estimate of the united solution set to the equation $F(\mathbf{a}, x) = \mathbf{b}$.

* The author's software which has "public domain" status can be downloaded from the server of Institute of computational technologies at `http://www.ict.nsc.ru/ftp/ict/interval`.

PROPOSITION 6.2 [99], [100]. *If a proper interval vector* $\mathbf{x}$ *is a formal solution to the equation*

$$F(\mathbf{a}, x) = \mathbf{b},$$

then $\mathbf{x} \subseteq \Xi_{tol}(F, \mathbf{a}, \mathbf{b})$*, i.e.,* $\mathbf{x}$ *is an inner interval estimate for the tolerable solution set to the equation* $F(\mathbf{a}, x) = \mathbf{b}$ *(or, in other words, a solution to the tolerance problem for the equation* $F(\mathbf{a}, x) = \mathbf{b}$*).*

PROPOSITION 6.3. *Let the mapping* F *be such that each of the variables* $a_1, a_2, \ldots, a_l$ *occurs only once and to the first power in at most one of the component expressions* $F_1, F_2, \ldots, F_m$*. If a proper interval vector* $\mathbf{x}$ *is a formal solution to the equation*

$$F(\text{dual } \mathbf{a}, x) = \text{dual } \mathbf{b},$$

then $\mathbf{x} \subseteq \Xi_{ctr}(F, \mathbf{a}, \mathbf{b})$*, i.e.,* $\mathbf{x}$ *is an inner interval estimate of the controllable solution set to the equation* $F(\mathbf{a}, x) = \mathbf{b}$*.*

As a simplest illustrative example, we consider the interval equation of two unknowns

$$[1, 2]\, x^2 + y^2 = [4, 10]. \tag{6.3}$$

One can readily see that its united solution set is the circle with the radius $\sqrt{10}$ and the center at the origin of coordinates, with the smaller ellipse deleted, which is depicted in Figure 9. The values $x = [0, 1]$ and $y = [2, 3]$ provide us with the formal solution to the dualization equation

$$[2, 1]\, x^2 + y^2 = [4, 10],$$

and Figure 9 shows that the interval vector $([0, 1], [2, 3])^\top$ really gives an inner interval estimate of the united solution set, even maximal with respect to inclusion order.

Notice that in Theorem 6.1 and the above propositions we could consider, with equal success, formal solutions of the inclusion

$$\mathbf{A}^{\mathbf{c}}\, x \subseteq \mathbf{b}^{\mathbf{c}}$$

and such like rather than those of the equation (6.1), which sometimes does make sense indeed. On the other hand, wider vector $\mathbf{x}$ leads to wider product $\mathbf{A}^{\mathbf{c}}\, \mathbf{x}$, so as it is intuitively clear that taking the equalities (6.1) instead of inclusions conduces to larger size of the formal solution. The results of Section 6.2 show that, to some extent, this is really so. The other reason why we consider mainly equations is that, in our particular case, they are amenable to more efficient computational procedures for finding their formal solutions.

The requirements Propositions 6.1–6.3 imposes on the occurrences of the E-uncertain interval parameters in the equations systems $F(\mathbf{a}, x) = \mathbf{b}$ are quite

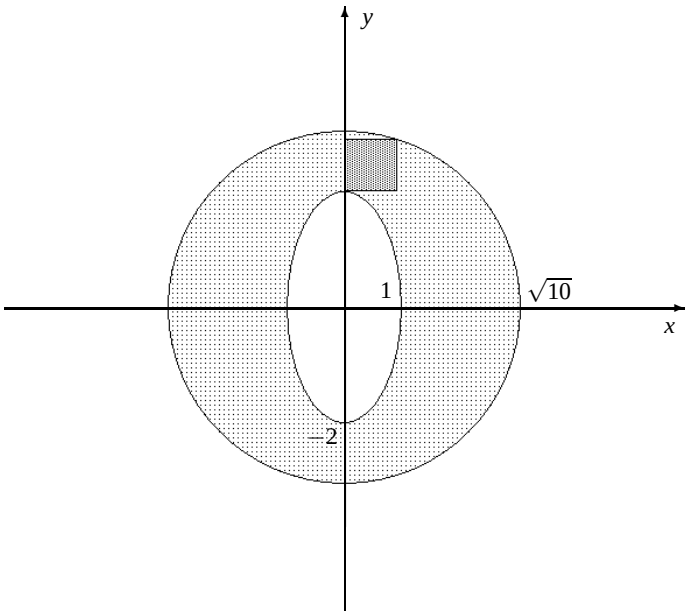

Figure 9. Inner estimation of the united solution set to the interval equation (6.3).

burdensome, so that a natural wish is to somehow relax them in order to extend our formal approach to the widest possible class of problems. We are able, to a certain extent, to do that by using the following trick, which we call "parameter freezing."

Notice that narrowing an interval of variations of any E-uncertain parameter may only lead to the AE-solution set also making more narrow with respect to inclusion. Additionally, an inner estimate of the narrowed solution set is an inner estimate of the solution set to the original interval system of equations too. Furthermore, when we squeeze the interval of an E-uncertain parameter to a *single point* ("freeze" the variation of this parameter), the corresponding uncertainty disappears. As a result, the structure of the original interval equations system $F(\mathbf{a}, x) = \mathbf{b}$ can get simpler, if the corresponding interval parameter had a multiple occurrence in this system.

Successively repeating the above "freezing" procedure, we may yield eliminating, from the original system, all the interval E-uncertain parameters that occur in more than one of the component expressions $F_1, F_2, \ldots, F_n$ thus bypassing the main obstacle of applying Propositions 6.1–6.3. Of course, for such a simplification of the interval system we have to pay by coarsening of our inner estimation, i.e. decreasing the size of the resulting estimate, and sometimes it can even become empty.

6.2. MAXIMALITY OF INNER ESTIMATES

In this subsection, we consider quality issues of the inner interval estimation by the formal approach, or, in other words, the question about the size of the inner interval estimate of the AE-solution sets. A remarkable feature of the formal approach, as applied to interval linear systems, is that it almost always gives us inner interval estimates of the generalized solution sets, which are *maximal with respect to inclusion*. This fact was first revealed by L. Kupriyanova [53] for the united solution set to interval linear systems. Afterward, S. Shary gave another form of that result and proved maximality of the estimates of the tolerable and controllable solution set obtained by the formal approach [106]. The following more general result was first found by S. Shary in [100]:

THEOREM 6.2. *If a proper interval vector is an inclusion-maximal formal solution to the dualization equation (6.1), it is also an inclusion-maximal interval vector contained in* $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$*, i.e., gives an inclusion-maximal solution to the inner estimation problem (4.3).*

In particular, if a proper formal solution to the dualization equation is unique, then it is an inclusion-maximal solution to the problem (4.3).

Proof. We need the following auxiliary representation: if $\mathbf{v}$ is a proper interval n-vector and $\mathbf{C}$ is an (arbitrary) interval $m \times n$-matrix, then

$$\mathbf{C}\cdot\mathbf{v} = \bigvee_{v\in\mathbf{v}} \mathbf{C}\cdot v. \tag{6.4}$$

Indeed, if $\mathbf{C}\cdot\mathbf{v} = ((\mathbf{C}\cdot\mathbf{v})_1, (\mathbf{C}\cdot\mathbf{v})_2, \ldots, (\mathbf{C}\cdot\mathbf{v})_m)^\top$, then, using (5.22) and distributivity (5.6) of addition with respect to the operation "$\vee$", we get

$$\begin{aligned}(\mathbf{C}\cdot\mathbf{v})_i &= \sum_{j=1}^{n} \mathbf{a}_{ij}\mathbf{v}_j = \sum_{j=1}^{n} \bigvee_{v_j\in\mathbf{v}_j} \mathbf{a}_{ij}v_j \\ &= \bigvee_{v_1\in\mathbf{v}_1} \bigvee_{v_2\in\mathbf{v}_2} \cdots \bigvee_{v_n\in\mathbf{v}_n} \sum_{j=1}^{n} \mathbf{a}_{ij}v_j \\ &= \bigvee_{v\in\mathbf{v}} \sum_{j=1}^{n} \mathbf{a}_{ij}v_j = \bigvee_{v\in\mathbf{v}} (\mathbf{C}\cdot v)_i.\end{aligned}$$

Now, let us turn to the proof of the theorem, which we will carry out *ad absurdum*. We denote the proper maximal formal solution of (6.1) by $\mathbf{x}$ and assume that, contrary to the assertion of the theorem, there exists a proper interval vector $\mathbf{y}$, such that

$$\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) \supseteq \mathbf{y} \supset \mathbf{x}.$$

Making use of inclusion monotonicity of the interval arithmetic operations from $\mathbb{KR}$ one obtains

$$\mathbf{A}^{\mathbf{c}}\cdot\mathbf{y} \supset \mathbf{A}^{\mathbf{c}}\cdot\mathbf{x} = \mathbf{b}^{\mathbf{c}},$$

the exact equality instead of inclusion being impossible due to the maximality of $\mathbf{x}$. Furthermore, the representation (6.4) results in

$$\bigvee_{y \in \mathbf{y}} \mathbf{A}^{\mathbf{c}} \cdot y \supset \mathbf{b}^{\mathbf{c}}, \tag{6.5}$$

and we can conclude that there must be

$$\mathbf{A}^{\mathbf{c}} \cdot \tilde{y} \not\subseteq \mathbf{b}^{\mathbf{c}}$$

for some (at least one) $\tilde{y} \in \mathbf{y}$. Otherwise, if we had $\mathbf{A}^{\mathbf{c}} \cdot y \subseteq \mathbf{b}^{\mathbf{c}}$ for all $y \in \mathbf{y}$, then the inclusion that is opposite to (6.5) would be valid. However, owing to Theorem 5.1, the relation (6.5) is equivalent to $\tilde{y} \notin \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, so $\mathbf{y} \not\subseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. □

An exhaustive study of the conditions under which our formal approach yields an inclusion-maximal inner estimates for AE-solution sets has been carried out in [97], [98]. The concluding result of these papers is

THEOREM 6.3. Irene's theorem. *If the interval matrix* $\mathbf{A}$ *has, in each column, at least one component that does not contain zero, then every proper formal solution to the dualization equation is a maximal inner interval estimate for the corresponding AE-solution set.*

If $\mathbf{A}^{\exists} = \mathbf{A}$*, then the above condition is even necessary for the interval estimate to be inclusion-maximal.*

The proof of this statement is not simple, both technically and conceptually, it requires deeper inquiring into the common inclusion monotonicity property of interval arithmetic operations, which is resulted in the introduction of the so-called *strong inclusion monotonicity*. Interesting results concerning the strong inclusion monotonicity is an important byproduct of the works [97], [98].

As an example, let us consider the formal solution of the interval systems

$$\begin{pmatrix} [\ 2,4] & [-2,1] \\ [-1,2] & [\ 4,2] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \qquad \text{and}$$
$$\begin{pmatrix} [2,\ 4] & [-2,1] \\ [2,-1] & [\ 4,2] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \tag{6.6}$$

which is the proper vector $(0, [-1,1])^{\top}$, and, in accordance with Theorem 6.3, it gives an inclusion-maximal inner estimate for $\binom{\forall\forall}{\forall\exists}\binom{\exists}{\exists}$-solution set and for $\binom{\forall\forall}{\exists\exists}\binom{\exists}{\exists}$-solution set of the model system (3.21). That it is actually so, one can make sure from Figure 4, but "flatness" of the estimate produced in the first coordinate may prove disadvantageous in practice. The rest of the AE-solution sets depicted at Figures 3 and 4 are estimated from inside by the formal approach more optimistically. In particular, for the interval linear systems

$$\begin{pmatrix} [\ 2,4] & [1,-2] \\ [-1,2] & [4,\ \ 2] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \qquad \text{and}$$
$$\begin{pmatrix} [2,\ 4] & [1,-2] \\ [2,-1] & [2,\ \ 4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$

the inner interval estimate yielded by the formal approach for the sets of $\binom{\forall}{\forall}\binom{\exists}{\exists}$-solutions and $\binom{\forall}{\exists}\binom{\exists}{\exists}$-solutions—the vector

$$\begin{pmatrix} \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \left[-\frac{1}{2}, \frac{1}{2}\right] \end{pmatrix}$$

—cover considerable parts of the corresponding sets indeed.

In connection with the last observation, the following important practical question arises. Namely, how can we influence upon the location and/or size of the interval solution to the inner estimation problem (4.1) obtained by our formal approach? We answer that in the next subsection of the work.

6.3. Correction of Inner Estimates

The most serious shortcoming of the formal approach to the inner estimation is that it does not allow to comprehensively examine the problem. If a solution to the dualization equation exists and it is proper, then everything is all right and we get the desired answer to the problem. Otherwise, if the dualization equation has no solutions, or, alternatively, the solutions do exist, but they are not proper, we can conclude nothing on whether $\Xi_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ is empty or not. For example, the one-dimensional equations

$$[-1, 1]\, x = [1, 2]$$

and

$$[-1, 1]\, x = [-1, 2]$$

do not have formal solutions at all, inasmuch as for any interval $\mathbf{x}$ the product $[-1, 1]\,\mathbf{x}$ is always a balanced interval, which is equal to $[-|\mathbf{x}|, |\mathbf{x}|]$ for proper $\mathbf{x}$ and $[-\langle \text{pro } \mathbf{x}\rangle, \langle \text{pro } \mathbf{x}\rangle]$ for improper $\mathbf{x}$. Since the right-hand sides of both above equations are not balanced, they cannot have formal solutions. Meanwhile, the first equation has empty tolerable solution set, but for the second one the tolerable solution set is nonempty: $\Xi_{tol} = [-1, 1]$.

Sometimes, either size or location of the interval solution to the inner estimation problem produced by the formal approach may prove unacceptable. A tool for correcting such situations is provided by the following

THEOREM 6.4. Lemma on "squeezing and inflating of parameters" for interval linear systems. *If an interval vector* $\mathbf{x}$ *is an inner estimate of an AE-solution set* $\Xi(\mathbf{A}^c, \mathbf{b}^c)$ *to an interval linear system that corresponds to the characteristic matrix* $\mathbf{A}^c$ *and right-hand side vector* $\mathbf{b}^c$*, then it is also an inner interval estimate of an AE-solution set* $\Xi(\tilde{\mathbf{A}}^c, \tilde{\mathbf{b}}^c)$ *to an interval linear system corresponding to the characteristic matrix* $\mathbf{A}^c$ *and right-hand side vector* $\mathbf{b}^c$ *such that*

$$\tilde{\mathbf{A}}^c \subseteq \mathbf{A}^c \qquad \textit{and} \qquad \mathbf{b}^c \subseteq \tilde{\mathbf{b}}^c.$$

Proof. For any point x, the membership $x \in \Xi(\mathbf{A}^c, \mathbf{b}^c)$ is equivalent to the inclusion

$$\mathbf{A}^c \cdot x \subseteq \mathbf{b}^c$$

in the complete interval arithmetic. Making use of the condition of Theorem and inclusion monotonicity property, we get

$$\tilde{\mathbf{A}}^c \cdot x \subseteq \mathbf{A}^c \cdot x \subseteq \mathbf{b}^c \subseteq \tilde{\mathbf{b}}^c,$$

to wit

$$\tilde{\mathbf{A}}^c \cdot x \subseteq \tilde{\mathbf{b}}^c,$$

which just means that the point x belongs to the solution set $\Xi(\tilde{\mathbf{A}}^c, \tilde{\mathbf{b}}^c)$ as well. Therefore, $\Xi(\mathbf{A}^c, \mathbf{b}^c) \subseteq \Xi(\tilde{\mathbf{A}}^c, \tilde{\mathbf{b}}^c)$, and the proof is complete. □

Let us turn to the previous example of the interval system $[-1,1]x = [-1,2]$ and its tolerable solution set. If the right-hand side of this system is squeezed to $[-1,1]$, then the resulted equation $[-1,1]x = [-1,1]$ becomes solvable, and its formal solution $[-1,1]$ coincides with the tolerable solution set of the system. Alternatively, we may inflate the left-hand side coefficient $[-1,1]$, which has A-uncertainty, to $[-1,2]$ and get the same result.

Notwithstanding the triviality of the proof for Theorem 6.4, the consequences of this result are extremely important for the computational practice. Specifically, if a client is dissatisfied by the results of the direct ("frontal") application of the formal approach to the initial interval system, it makes good sense to try it with an auxiliary interval system that has "squeezed" intervals corresponding to A-uncertainty and "expanded" intervals corresponding to E-uncertainty. You can get something better in location and/or size, "more solid" in particular. Sometimes, this simple technique enables one to compute inner interval estimates of the solution sets even for the situations, when the dualization equation that corresponds to the initial system does not have proper solutions at all.

Let us consider specific examples. For the interval linear system

$$\begin{pmatrix} [\ 2,4] & [-1,1] \\ [-1,1] & [\ 2,4] \end{pmatrix} x = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix} \tag{6.7}$$

from [71], [69], both $\binom{\exists\exists}{\exists\exists}\binom{\exists}{\exists}$-solution set (i.e., the united solution set) and $\binom{\exists\forall}{\exists\exists}\binom{\exists}{\exists}$-solution set have butterfly-shaped forms drawn in Figure 10. When seeking inner interval estimates for them, the direct use of Theorem 6.1, i.e. computing formal solutions to the equations of the type (6.1), leads in both cases to $([-1.5, 1.5], 0)^\top$, which is not very successful because of the second degenerate component.

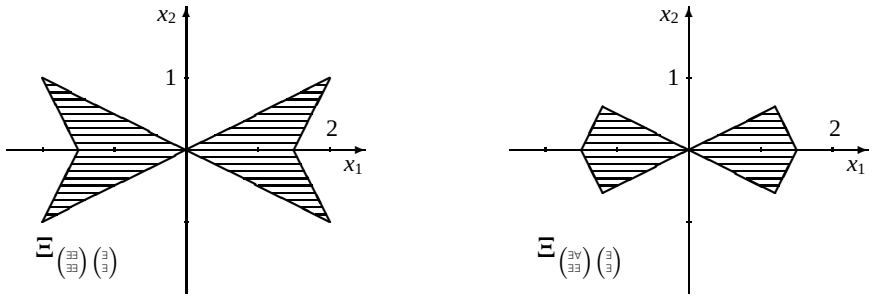

Figure 10. "Almost disconnected" solution sets to the interval equation (6.7).

If we try with the same matrix and a squeezed right-hand side, say, $[1,3]$, rather than $[-3,3]$, in the first component of the right-hand side (since the zero in the second component cannot be changed), we get proper formal solutions

$$\begin{pmatrix} \left[\frac{1}{4},\frac{3}{2}\right] \\ \left[-\frac{1}{8},\frac{1}{8}\right] \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \left[\frac{2}{7},\frac{10}{7}\right] \\ \left[-\frac{1}{7},\frac{1}{7}\right] \end{pmatrix}$$

for the corresponding dualization equations. The above interval vectors have nonempty interiors and cover larger parts of the solution sets, which sometimes may appear to be more advantageous for the users. Of course, we may try to vary the first components of the right-hand side vector in another manner, combining it with varying the matrix as well.

One more example. Suppose that, for the interval linear system

$$\begin{pmatrix} [2,3] & 1 \\ 1 & [2,3] \end{pmatrix} x = \begin{pmatrix} [-5,5] \\ 0 \end{pmatrix}, \tag{6.8}$$

we are to find an inner interval estimate of the united solution set depicted at Figure 11. The direct application of Theorem 6.1 and computing the formal solution to the dualization equation for (6.8) leads to the interval vector $([-3,3],[1,-1])^\top$, which has the second component improper and thus cannot be interpreted as an inner interval estimate. However, it is quite clear that the united solution set to (6.8) is nonempty and even its interior is nonempty as well!

The reason why the algebraic approach fails is that the solution set for the system considered is a butterfly-shaped region with the origin of coordinates being a "singular" point. In such situations, one should not expect to get a solution "at one fell swoop," since an inner estimate for the solution set may not, in principle, be adequately represented by a single interval that covers all parts of the solution set (which belong to different orthants). In general, AE-solution sets are proved to be a complex non-convex set, but in the situation we deal with the solution set is extremely non-convex, almost disconnected. More precisely, it is constituted of the two components that touch in the only point, the origin of coordinates. Formal

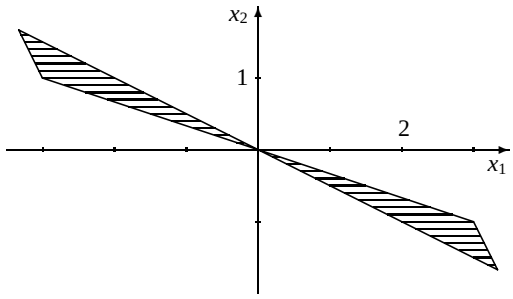

Figure 11. "Almost disconnected" united solution set to the interval linear system (6.8).

approach "tries" to find one interval vector that would cover all these weakly connected regions and, naturally, fails.

How to cope with our troubles?

One had better seek the answer to the inner estimation problem in the form of the union of several intervals (depending on the dimension of the system). Specifically, our prescript is "divide and conquer," keeping in mind the proposition about system squeezing.

The major evident reasons that cause bad shape of the solution set (and consequently bad results produced by formal approach) are

- "thin" zeros in the right-hand side,
- zero-containing intervals in the right-hand side, and (the least important)
- zero-containing interval entries in the matrix.

There are less evident reasons, but we skip them for simplicity. Thus,

- "divide" means "subdivide your system to squeezed systems," to get rid of zero-containing components combined with zeros. That must result in a simplification of the solution set, and the "butterfly-shape" will vanish,
- "conquer" means "solve inner estimation problems for them separately," and afterward you can get something (more or less) suitable by gathering the separate answers.

If, in the interval system (6.8), we change the characteristic right-hand vector to $([5,4],0)^\top$, such that $([5,4],0)^\top \subseteq ([-5,5],0)^\top$, retaining the characteristic matrix the same. Having computed the algebraic solution to the dualization equation

$$\begin{pmatrix} [3,2] & 1 \\ 1 & [3,2] \end{pmatrix} x = \begin{pmatrix} [5,4] \\ 0 \end{pmatrix}, \tag{6.9}$$

we get a solid interval vector $([2,2.4],[-1,-0.8])^\top$ as a required inner estimate of the united solution set for (6.8). Notice that to obtain such an apt estimate we had to change the interval uncertainty type in the second component of the right-hand

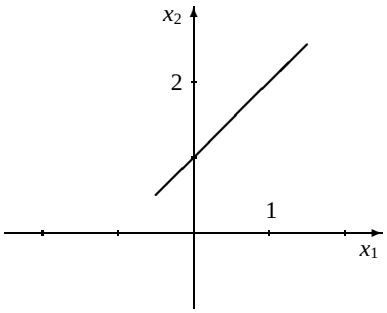

Figure 12. Nonsolid united solution set to the interval linear system (6.10).

vector of the auxiliary system (6.9): it has become different from that in the original system.

Of course, there are situations when the interior of the solution set is empty so that a "good" inner interval estimate can be found in no way. Let us consider, for instance, the interval linear system

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x = \begin{pmatrix} [0,4] \\ 1 \end{pmatrix}. \tag{6.10}$$

Its united solution set is a straight line segment depicted at Figure 12, for which a solid inner estimate does not exist in principle. We have to be able to discern such cases in practice.

6.4. Existence and Uniqueness of Formal Solutions

DEFINITION 6.2 [103]. Given a square matrix $Q \in \mathbb{R}^{n\times n}$, we put

$$Q^{\sim} := \left(\begin{array}{c|c} Q^+ & Q^- \\ \hline Q^- & Q^+ \end{array}\right), \tag{6.11}$$

where $n \times n$-submatrices $Q^+ = (q_{ij}^+)$ and $Q^- = (q_{ij}^-)$ are positive and negative parts of Q, i.e. the matrices composed of the positive and negative parts of the entries of Q respectively. The matrix $Q^{\sim} \in \mathbb{R}^{2n\times 2n}$ is said to be *concomitant matrix* to Q.

THEOREM 6.5. *For a square matrix $Q \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) $Q\mathbf{x} = 0$ *in the interval space* $\mathbb{KR}^n$ *if and only if* $\mathbf{x} = 0$*;*

(ii) *the matrix* $Q^{\sim} \in \mathbb{R}^{2n\times 2n}$*, concomitant to* Q*, is regular;*

(iii) *both the matrix* Q *itself and its module* $|Q|$ *(i.e., the matrix made up of the moduli of the entries) are regular matrices.*

Proof. The equivalence (i)⇔(ii) is substantiated e.g. in [100], [103], [109].

To prove the equivalence (ii)⇔(iii), let us transform the concomitant matrix $Q^{\sim}$ as follows [109]. We add the first row of $Q^{\sim}$ to its $(n+1)$-st row, the second row to the $(n+2)$-nd one, etc., up to the n-th row which we add to the $2n$-th one. Insofar as

$$q^{+} + q^{-} = |q|$$

for each real q, we get the following block $2n \times 2n$-matrix

$$\begin{pmatrix} Q^{+} & Q^{-} \\ |Q| & |Q| \end{pmatrix} \tag{6.12}$$

as the result of our transformations. Next, we subtract the $(n+1)$-th column of the matrix (6.12) from its first column, the $(n+2)$-nd column from the second one, etc., up to the $2n$-th column which we subtract from the n-th one. Insofar as

$$q^{+} - q^{-} = q$$

for each real q, we get the following block-triangular $2n \times 2n$-matrix

$$\begin{pmatrix} Q & Q^{-} \\ 0 & |Q| \end{pmatrix}. \tag{6.13}$$

As is known from linear algebra, our transformations do not change the property of a matrix to be either singular or regular, so that the matrix (6.13) is singular or regular simultaneously with the concomitant matrix $Q^{\sim}$. At the same time, the determinant of (6.13) is equal to the product of the determinants of Q and $|Q|$. □

DEFINITION 6.3. A square matrix $Q \in \mathbb{R}^{n\times n}$ satisfying any (and, therefore, all) of the equivalent conditions listed in Theorem 6.5 is called *completely regular (completely nonsingular).*[⋆]

For example, the unit matrix is completely regular, while the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

being regular in the usual sense, is not completely regular. It is apparent that *a priori* a matrix is not completely regular if it is singular. Also, every nonnegative regular matrix is completely regular.

The main results of this subsection are local existence and uniqueness theorems for formal solutions of interval linear equations of the form (3.14).

THEOREM 6.6. *If the interval matrix* $\mathbf{C} \in \mathbb{KR}^{n\times n}$ *is sufficiently narrow (i.e., if* $\|\mathrm{rad}\ \mathbf{C}\|$ *is sufficiently small) and* pro **C***—proper projection of* **C***—contains a completely regular point matrix, then the equations system*

[⋆] The author called such matrices *ι-nonsingular* in the previous papers [103], [105], [106], [113].

$$\mathbf{C}x = \mathbf{d} \tag{6.2}$$

has a formal solution in Kaucher complete interval arithmetic for every $\mathbf{d} \in \mathbb{KR}^n$.

One can find the complete proof of this fact, e.g., in [109]. It is conducted by topological methods (see, e.g., [77]), using the theory of rotation of vector fields (or equivalent theory of topological degree of mappings).

Let us turn to the uniqueness issues of the formal solutions to square interval linear systems of the form (6.2) in Kaucher complete interval arithmetic. The questions is completely solved by the "immersion theory" of [100], [103], [105], [109] for the systems (6.2) with point matrices. Namely, the interval system

$$Cx = \mathbf{d}$$

with a completely regular matrix C has a unique formal solution for every right-hand side vector $\mathbf{d} \in \mathbb{KR}^n$. As for the interval linear systems (6.2) with essentially interval matrices $\mathbf{C}$, the uniqueness of their formal solutions has been relatively little studied for the time being.

DEFINITION 6.4. *Mignitude* $\langle \mathbf{a} \rangle$ of a proper interval $\mathbf{a}$ is the least distance of the points of $\mathbf{a}$ to zero, i.e.

$$\langle \mathbf{a} \rangle := \begin{cases} \min\{|\underline{\mathbf{a}}|, |\overline{\mathbf{a}}|\}, & \text{if } \mathbf{a} \not\ni 0, \\ 0, & \text{if } \mathbf{a} \ni 0. \end{cases}$$

DEFINITION 6.5. A proper interval matrix $\mathbf{A} = (\mathbf{a}_{ij})$ is called *strictly diagonally dominant* if it satisfies

$$\langle \mathbf{a}_{ii} \rangle > \sum_{k \neq i} |\mathbf{a}_{ik}| \qquad \text{for } i = 1, 2, \ldots, n. \tag{6.14}$$

A simple particular result that may sometimes prove helpful:

THEOREM 6.7 [52], [65], [96]. *For the interval linear system*

$$\mathbf{C}x = \mathbf{d} \tag{6.2}$$

with $\mathbf{C} \in \mathbb{KR}^{n \times n}$ *and* $\mathbf{d} \in \mathbb{KR}^n$, *the formal solution exists and is unique provided that the proper projection* pro $\mathbf{C}$ *is strictly diagonally dominant.*

Proof. We introduce the following notation:

$\mathbf{D}$ is the diagonal matrix diag $\{\mathbf{c}_{11}, \mathbf{c}_{22}, \ldots, \mathbf{c}_{nn}\}$,

$\mathbf{E}$ is the matrix obtained from $\mathbf{C}$ by nullifying its diagonal entries.

Therefore, $\mathbf{C} = \mathbf{D} + \mathbf{E}$, and the formal solution to the system (6.2) obviously coincides with that of the system

$$\mathbf{D}x + \mathbf{E}x = \mathbf{d},$$

which in its turn is equivalent to

$$\mathbf{D}x = \mathbf{d} \ominus \mathbf{E}x.$$

We can thus arrange the iterations

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}(\mathbf{d} \ominus \mathbf{E}\mathbf{x}^{(k)})$$

with $\mathbf{D}^{-1} = \operatorname{diag}\{\mathbf{c}_{11}^{-1}, \mathbf{c}_{22}^{-1}, \ldots, \mathbf{c}_{nn}^{-1}\}$, and they converge to a unique fixed point of the map

$$\mathbf{x} \mapsto \mathbf{D}^{-1}(\mathbf{d} \ominus \mathbf{E}\mathbf{x})$$

due to the strict diagonal dominance in $\mathbf{C}$. □

7. Outer Estimation of the Solution Sets

In this section, we present several approaches to outer interval estimation of the AE-solution sets to interval systems of equations. They are quite unequal both with respect to their computational complexity and the quality of the answers they produce. These are, on the one hand, exponentially complex methods of Section 7.1 and Section 7.5 intended for the computation of the outer estimates which are optimal or have guaranteed accuracy. On the other hand, the methods from Section 7.2 and Section 7.3 may yield substantial overestimation of the solution sets, but they are not so labor consuming. It should be remarked that a technique for outer interval estimation of the AE-solution sets, which is quite similar in idea to our "formal approach" of Section 7.2, has been recently proposed by M. Sainz and E. Gardeñes in [95].

7.1. Exhaustive Search Like Methods

Let us turn to the outer estimation problem in the componentwise form (4.5). As we have established in Theorem 3.6, the intersection of the AE-solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ to an interval linear system with every orthant $\mathcal{O}$ of the space $\mathbb{R}^n$ is a convex polyhedral set (which may be empty). Computing the values

$$\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \cap \mathcal{O}\}, \qquad v = 1, 2, \ldots, n, \tag{7.1}$$

amounts therefore to a linear programming problems that may be efficiently solved, e.g., by the widely known and well developed simplex-method. Further, we need to look over all the orthants of the space $\mathbb{R}^n$ and to pick out the smallest among the computed values (7.1). In the general case, this approach is of low practical significance due to the enormous growth of its computational complexity with the dimension of the problem. Still, for small dimensions and for some special types of the interval linear systems (for instance, when it is *a priori* known that the solution set is situated in a limited number of orthants), the above exhaustive search can

be successfully applied to the practical solution of the outer estimation problem (4.4)–(4.5). For the particular case of the united solution set, this kind of method has been considered earlier in the works [17], [75].

We are going to write down, similar to what has been done in [17], [75], the so-called *canonical form* of the linear programming problem we have to solve to find the values (7.1). Let us use Rohn characterization (Theorem 3.5), which gives the description of the points from the AE-solution set through linear inequalities with moduli:

the membership $x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ is equivalent to the componentwise inequality $|(\operatorname{mid} \mathbf{A}) \cdot x - \operatorname{mid} \mathbf{b}| \le (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \cdot |x| + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall})$,

or

$$\begin{cases} \operatorname{mid} \mathbf{A} \cdot x - \operatorname{mid} \mathbf{b} \;\le\; (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \cdot |x| + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall}), \\ -\operatorname{mid} \mathbf{A} \cdot x + \operatorname{mid} \mathbf{b} \;\le\; (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \cdot |x| + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall}), \end{cases}$$

which, in its turn, is equivalent to the system

$$\begin{cases} \operatorname{mid} \mathbf{A} \cdot \operatorname{diag}\{\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n\} \cdot |x| - (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \cdot |x| \\ \qquad\qquad \le \quad \operatorname{mid} \mathbf{b} + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall}), \\ -\operatorname{mid} \mathbf{A} \cdot \operatorname{diag}\{\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n\} \cdot |x| - (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \cdot |x| \\ \qquad\qquad \le \; -\operatorname{mid} \mathbf{b} + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall}), \end{cases}$$

where $\operatorname{diag}\{\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n\}$ is the diagonal matrix with $\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n$ along its main diagonal. Also, the right-hand sides of these inequalities may be further simplified taking into account the definition of the characteristic right-hand side vector $\mathbf{b}^{\mathsf{c}}$:

$$\begin{aligned} \operatorname{mid} \mathbf{b} + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall}) &= \overline{\mathbf{b}^{\mathsf{c}}}, \\ -\operatorname{mid} \mathbf{b} + (\operatorname{rad} \mathbf{b}^{\exists} - \operatorname{rad} \mathbf{b}^{\forall}) &= \overline{(-\mathbf{b}^{\mathsf{c}})}. \end{aligned}$$

Let y be the vector of the absolute values of x, i.e. $y_i = |x_i|$, $i = 1, 2, \ldots, n$, and

$$S = \operatorname{diag}\{s_1, s_2, \ldots, s_n\}, \qquad s_i = \operatorname{sgn} x_i = \pm 1,$$

be the diagonal matrix formed by the signs of the interior points of the orthant $\mathcal{O}$ under consideration, i.e. $x = Sy$ for $x \in \mathcal{O}$. Then the condition

$$x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \cap \mathcal{O}$$

is satisfied if and only if there exists $y \in \mathbb{R}^n$ such that

$$\begin{cases} \begin{pmatrix} \operatorname{mid} \mathbf{A} \cdot S - (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \\ -\operatorname{mid} \mathbf{A} \cdot S - (\operatorname{rad} \mathbf{A}^{\exists} - \operatorname{rad} \mathbf{A}^{\forall}) \end{pmatrix} y \;\le\; \begin{pmatrix} \overline{\mathbf{b}^{\mathsf{c}}} \\ \overline{(-\mathbf{b}^{\mathsf{c}})} \end{pmatrix}, \\ y \;\ge\; 0. \end{cases} \tag{7.2}$$

Table 3. Passive exhaustive-search algorithm for the outer estimation of the AE-solution sets to interval linear systems.

Input

An interval linear system $\mathbf{A}x = \mathbf{b}$.
An integer index $v \in \{1, 2, \ldots, n\}$.

Output

The exact lower estimate $y = \min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ along the v-th coordinate.

Algorithm

$y \leftarrow +\infty$;
`DO FOR` $i = 0$ `TO` $2^n - 1$
    solve the linear programming problem (7.2)–(7.3)
        computing $z \leftarrow \min\{x_v \mid x \in \Xi_{\alpha\beta} \cap \mathcal{O}_i\}$;
    `IF` ($y > z$) $y \leftarrow z$;
`END DO`

Therefore, the value $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \cap \mathcal{O}\}$ is the solution to the linear programming problem with the constraints (7.2) and the objective function

$$c^\top y, \qquad c^\top = (0, \ldots, 0, s_v, 0, \ldots, 0) \in \mathbb{R}^n, \tag{7.3}$$

to be minimized.

Each orthant of the space $\mathbb{R}^n$ is completely determined by indicating the signs of its interior points and, for the algorithmic purposes, it will be convenient for us to enumerate all the orthants of $\mathbb{R}^n$ by the integers from 0 to $2^n - 1$. That can be done, for example, as follows: every orthant gets an n-digit binary number obtained from the component signs set through replacing "$-$" for 0 and "+" for 1. Overall, the exact outer estimate of the value $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ can be found by the algorithm whose pseudocode is presented in Table 3 (where "$\leftarrow$" means assigning operator).

DEFINITION 7.1. An algorithm is called *passive*, if performing its every step (information computation) does not make use of the information obtained at the preceding steps.

An algorithm is called *adaptive*, if performing its every step (information computation) does make use (in this or that form) of the information from the preceding steps.

The passive algorithms are thus the algorithms with rigidly predetermined computational scenarios that do not depend on individual features of the problems under solution. On the contrary, the adaptive algorithms enable us to flexibly adapt the solution process to each specific problem. Therefore, such algorithms are more

preferable in the computational practice, all the other factors being equal.* As we can see, the technique presented in this subsection for the outer estimation of the AE-solution sets to interval linear systems turns out to be only passive, which is one of its major drawbacks.

A natural and promising way to modify the above approach is to examine only the orthants whose intersections with the solution set are *a priori* nonempty, which is to be revealed in a special procedure not connected with the solution of the linear programming problem (7.2)–(7.3). For the united solution set, this idea has been first advanced and implemented by C. Jansson [38], its extension to general AE-solution sets being quite evident. Checking nonemptyness requires additional labor, but the efforts will be rewarded by much far less average execution complexity as compared with that of the passive exhaustive-search algorithm of Table 3. Moreover, the new modified algorithm becomes *adaptive*.

7.2. Formal Approach

In this subsection, we demonstrate how the problem of outer interval estimation of the AE-solution sets can be reduced to the problem of computing formal solutions to a special interval system of equations. The whole idea is thus very much alike to that of the "formal approach," which we have applied for inner interval estimation in Section 6. For this reason, we shall also refer to this outer estimation technique as "formal approach". On the other hand, in Theorems 7.2–7.4, one can easily recognize the analogs of the classical results on outer estimation in the interval linear systems from [2], [3].

Below, we shall need yet another characterization of the AE-solution sets to the interval linear systems with square matrices, which has the so called *fixed-point form*, and the starting point of our consideration is the result of Theorem 5.1:

$$x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \qquad \Longleftrightarrow \qquad \mathbf{A}^{\mathsf{c}}\, x \subseteq \mathbf{b}^{\mathsf{c}}.$$

Adding $(x \ominus \mathbf{A}^{\mathsf{c}}x)$ to both sides of the above inclusion, we get the equivalent relation

$$x \subseteq x + \operatorname{opp}(\mathbf{A}^{\mathsf{c}}x) + \mathbf{b}^{\mathsf{c}}.$$

But $\operatorname{opp}(\mathbf{A}^{\mathsf{c}}x) = (\operatorname{opp} \mathbf{A}^{\mathsf{c}})\, x$ for noninterval x. Therefore,

$$x \subseteq x + (\operatorname{opp} \mathbf{A}^{\mathsf{c}})\, x + \mathbf{b}^{\mathsf{c}},$$

and, again taking into account the fact that x is a point, we can avail ourselves of the distributivity and factor out the unknown variable x. Overall,

$$x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \quad \Longleftrightarrow \quad x \subseteq (I \ominus \mathbf{A}^{\mathsf{c}})\, x + \mathbf{b}^{\mathsf{c}}.$$

* In a sense, the distinction between the passive/adaptive algorithms corresponds to the contrast between the program and position-based controls of a dynamic object mentioned at page 331.

It worthwhile to note that, for $x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \neq \emptyset$, our reasoning imply that $(I \ominus \mathbf{A}^{\mathfrak{c}})\, x + \mathbf{b}^{\mathfrak{c}}$ is a proper interval vector.

We thus arrive at

THEOREM 7.1. *For the square interval linear system* $\mathbf{A}x = \mathbf{b}$, *a point* $x \in \mathbb{R}^n$ *belongs to the solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *if and only if*

$$x \subseteq (I \ominus \mathbf{A}^{\mathfrak{c}})\, x + \mathbf{b}^{\mathfrak{c}}$$

in Kaucher complete interval arithmetic.

Our next result is

THEOREM 7.2. *Let an interval matrix* $\mathbf{C} \in \mathbb{KR}^{n \times n}$ *be such that the spectral radius* $\rho(|\mathbf{C}|)$ *of the matrix made up of the moduli of its elements is less than* 1. *Then a formal solution to the interval linear system*

$$x = \mathbf{C}x + \mathbf{d} \tag{7.4}$$

exists and is unique for any interval vector $\mathbf{d} \in \mathbb{KR}^n$.

Proof. In the situation under study

$$\mathrm{Dist}\,(\mathbf{C}\mathbf{x}' + \mathbf{d},\ \mathbf{C}\mathbf{x}'' + \mathbf{d}) = \mathrm{Dist}\,(\mathbf{C}\mathbf{x}',\ \mathbf{C}\mathbf{x}'') \leq |\mathbf{C}| \cdot \mathrm{Dist}\,(\mathbf{x}', \mathbf{x}'')$$

for any vectors $\mathbf{d}, \mathbf{x}', \mathbf{x}'' \in \mathbb{KR}^n$ and for the pseudometric Dist introduced by (5.19). If the spectral radius of the matrix $|\mathbf{C}|$ is less than one, we can make use of the finite-dimensional version of Shröder fixed-point theorem (see, e.g., [2], [16], [69], [77]). That is, the mapping $\mathbb{KR}^n \to \mathbb{KR}^n$ which acts as

$$\mathbf{x} \mapsto \mathbf{C}\mathbf{x} + \mathbf{d},$$

is contracting with respect to the pseudometric Dist, and thus has the only fixed point, which is the formal solution to the interval linear system (7.4). □

THEOREM 7.3. *Let, for an interval linear system* $\mathbf{A}x = \mathbf{b}$, *an AE-solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *be nonempty, while* $\mathbf{A}^{\mathfrak{c}}$ *and* $\mathbf{b}^{\mathfrak{c}}$ *are the corresponding characteristic matrix and right-hand side. If*

$$\rho(|I \ominus \mathbf{A}^{\mathfrak{c}}|) < 1, \tag{7.5}$$

then the formal solution of the interval linear system

$$x = (I \ominus \mathbf{A}^{\mathfrak{c}})\, x + \mathbf{b}^{\mathfrak{c}} \tag{7.6}$$

(which exists and is unique by virtue of Theorem 7.2) is a proper interval vector that contains the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

Proof. Assuming $\mathbf{x}^*$ to be a formal solution to the interval linear system (7.6), we are going to show that $\tilde{x} \in \mathbf{x}^*$ providing that $\tilde{x} \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

In view of Theorem 7.1, the membership $\tilde{x} \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ is equivalent to the inclusion

$$\tilde{x} \subseteq (I \ominus \mathbf{A}^{\mathsf{c}})\,\tilde{x} + \mathbf{b}^{\mathsf{c}}. \tag{7.7}$$

Let us organize an iteration process in $\mathbb{KR}^n$ according to the following formulas

$$\mathbf{x}^{(0)} \leftarrow \tilde{x}, \tag{7.8}$$

$$\mathbf{x}^{(k+1)} \leftarrow (I \ominus \mathbf{A}^{\mathsf{c}})\,\mathbf{x}^{(k)} + \mathbf{b}^{\mathsf{c}}. \tag{7.9}$$

Using induction, it is not hard to prove that all the consecutive interval vectors generated by the process contain $\tilde{x}$. Indeed, that holds true for $\mathbf{x}^{(0)}$ according to our construction. If $\tilde{x} \in \mathbf{x}^{(k)}$, then by virtue of (7.7) and the inclusion monotonicity of arithmetic operations in $\mathbb{KR}$

$$\tilde{x} \subseteq (I \ominus \mathbf{A}^{\mathsf{c}})\,\tilde{x} + \mathbf{b}^{\mathsf{c}} \subseteq (I \ominus \mathbf{A}^{\mathsf{c}})\,\mathbf{x}^{(k)} + \mathbf{b}^{\mathsf{c}} = \mathbf{x}^{(k+1)}. \tag{7.10}$$

Hence, $\tilde{x} \in \mathbf{x}^{(k)}$ for any natural number k.

Next, the condition $\rho(|\, I \ominus \mathbf{A}^{\mathsf{c}}|) < 1$ implies convergence of the iteration process defined by the formulas (7.8)–(7.9): Shröder fixed-point theorem (see [2], [16], [69], [77]) works again! So, the sequence $\mathbf{x}^{(k)}$ converges to the unique formal solution of the equation (7.6), that is, to $\mathbf{x}^*$. Since the membership $x \in \mathbf{x}^{(k)}$ is equivalent to the system of $2n$ nonstrict inequalities, it must hold after passing to the limit:

$$\tilde{x} \in \lim_{k \to \infty} \mathbf{x}^{(k)} = \mathbf{x}^*,$$

which completes the proof. □

We will also call the interval equations of the form (7.6) *dualization equations* that correspond to this or that AE-solution set of the interval linear system (1.3)–(1.4), stressing, if necessary, that the outer estimation mode is meant.

Now, it is time to remind the following

DEFINITION 7.2 (see [9], [69]). A matrix $A \in \mathbb{R}^{n\times n}$ is termed an *M-matrix*, if it satisfies any one of the following equivalent conditions:

(i) $A = sI - P$, where P is a nonnegative matrix and $s > \rho(P)$;

(ii) off-diagonal entries of the matrix A are non-positive and $A^{-1} \geq 0$;

(iii) off-diagonal entries of the matrix A are non-positive and there exists a vector $u > 0$ such that $Au > 0$;

(iv) …, etc.*

* For instance, A. Berman and R. Plemmons [9] list 50 conditions equivalent to the statement "the matrix A is an M-matrix."

DEFINITION 7.3 [6]. An interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is termed an *interval M-matrix* if every point matrix $A \in \mathbf{A}$ is an M-matrix.

There holds

THEOREM 7.4 [109]. *Let, for an interval linear system* $\mathbf{A}x = \mathbf{b}$*, an AE-solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *be nonempty, while* $\mathbf{A}^{\mathrm{c}}$ *and* $\mathbf{b}^{\mathrm{c}}$ *are the corresponding characteristic matrix and right-hand side. If* $\mathbf{A}$ *is an interval* M*-matrix and no component of* $\mathbf{b}$ *contain zero in its interior, then the formal solution of the interval linear system*

$$x = (I \ominus \mathbf{A}^{\mathrm{c}})\, x + \mathbf{b}^{\mathrm{c}}$$

exists, is unique and provides us with the interval hull of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$*, i.e., the best possible solution to the outer estimation problem (4.4).*

As an illustrative example, we can compute an enclosure of $\binom{\exists}{\exists}\binom{\forall}{\exists}$-solution set to the interval linear system (6.7). First, let us scale the system through multiplying both the matrix and right-hand side by $\frac{1}{4}$. We get

$$\begin{pmatrix} \left[\frac{1}{2}, 1\right] & \left[-\frac{1}{4}, \frac{1}{4}\right] \\ \left[-\frac{1}{4}, \frac{1}{4}\right] & \left[\frac{1}{2}, 1\right] \end{pmatrix} x = \begin{pmatrix} \left[-\frac{3}{4}, \frac{3}{4}\right] \\ 0 \end{pmatrix}. \tag{7.11}$$

The corresponding dualization equation (7.6) to enclose the $\binom{\exists}{\exists}\binom{\forall}{\exists}$-solution set for (7.11) has the form

$$x = \begin{pmatrix} \left[0, \frac{1}{2}\right] & \left[\frac{1}{4}, -\frac{1}{4}\right] \\ \left[-\frac{1}{4}, \frac{1}{4}\right] & \left[0, \frac{1}{2}\right] \end{pmatrix} x + \begin{pmatrix} \left[-\frac{3}{4}, \frac{3}{4}\right] \\ 0 \end{pmatrix},$$

and its formal solution is (check that!) the interval vector

$$\begin{pmatrix} \left[-\frac{3}{2}, \frac{3}{2}\right] \\ \left[-\frac{3}{4}, \frac{3}{4}\right] \end{pmatrix}.$$

It really includes the $\binom{\exists}{\exists}\binom{\forall}{\exists}$-solution set of the interval linear systems (6.7) and (7.11), as one can easily see from Figure 10.

We conclude the subsection with a commentary on the practical implementation of the above formal approach for outer interval estimation of the solution sets. Theorems 7.2–7.3 give, as a matter of fact, a theoretical foundation for constructing stationary single-step iterative techniques based on Schröder fixed-point theorem. Namely, under conditions of Theorem 7.3 we can organize iterating according to the formula (7.9) (or some its modification), which will really converge to an enclosure of the AE-solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ from any starting approximation $\mathbf{x}^{(0)}$. In doing so, the most convenient choice for $\mathbf{x}^{(0)}$ is a vector that is guaranteed to contain

$\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$, since then (7.10) implies that every iteration $\mathbf{x}^{(k)}$ contains the solution set under estimation too. For instance, we can take, as $\mathbf{x}^{(0)}$, an enclosure of the united solution set, its computation being a well-elaborate numerical problem (see, e.g., [2], [32], [45], [69], [87]).

Another opportunity to find the desired formal solution to (7.6) is *subdifferential Newton method* [103], [109] whose applicability is currently substantiated for the equations (7.6) with the matrices $\mathbf{A}^{\mathbf{c}}$ having either all proper or all improper entries along each row. However, numerical experiments show that the method works well for general interval linear systems, when proper and improper entries in $\mathbf{A}^{\mathbf{c}}$ are arbitrarily mixed (although it is *quasidifferential* Newton method, not subdifferential, in such cases).*

Finally, the requirement

$$\rho(|I \ominus \mathbf{A}^{\mathbf{c}}|) < 1 \tag{7.5}$$

that an interval linear system is amenable to our formal approach appears to be quite burdensome. Below, in Section 7.4, we will discuss a way to partially overcome the inequality (7.5) through the so-called *preconditioning*.

7.3. Interval Gauss-Seidel Method

Interval Gauss-Seidel method is known to be one of the most efficient and popular algorithms for the computation of the outer interval estimates (enclosures) of the united solution set to interval linear systems of equations. It is usually used after preliminary *preconditioning* of the interval systems (see, e.g., [45], [69]). The purpose of this section is to adapt the interval Gauss-Seidel iteration to the problems of outer interval estimation of the generalized solution sets to interval linear systems. Below, we suppose that the interval matrix $\mathbf{A}$ is nonsingular, i.e., that all the point matrices $A \in \mathbf{A}$ are nonsingular. One can achieve then, after suitable permutation of the equations (matrix rows), that the diagonal entries $\mathbf{a}_{ii}$, $i = 1, 2, \ldots, n$, do not contain zeros.

The basis of the point Gauss-Seidel method is writing out the system of equations $Ax = b$ in the explicit componentwise manner

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \ldots, n,$$

and further solving the i-th equation with respect to x_i assuming that $a_{ii} \neq 0$. To construct the interval method, we shall act in a similar way.

Let us make use of the characterization of the AE-solution sets to the interval linear systems presented by Theorem 5.1:

$$x \in \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) \qquad \Longleftrightarrow \qquad \mathbf{A}^{\mathbf{c}}x \subseteq \mathbf{b}^{\mathbf{c}}. \tag{5.26}$$

* Again, one can find some of the author's "public domain" implementations of the methods under discussion at `http://www.ict.nsc.ru/ftp/ict/interval`.

Breaking down the inclusion (5.26) componentwise, we get

$$\sum_{j=1}^{n} \mathbf{a}_{ij}^{\mathfrak{c}} x_j \subseteq \mathbf{b}_i^{\mathfrak{c}}, \qquad i = 1, \ldots, n,$$

which is equivalent to

$$\mathbf{a}_{ii}^{\mathfrak{c}} x_i \subseteq \operatorname{opp}\left(\sum_{j \neq i} \mathbf{a}_{ij}^{\mathfrak{c}} x_j\right) + \mathbf{b}_i^{\mathfrak{c}}, \qquad i = 1, \ldots, n.$$

Suppose that we are already given an interval vector $\mathbf{x}$ containing the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, i.e. $\mathbf{x} \supseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. Then, for $x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ the following chain of relations should be valid

$$\begin{aligned}
x_i \;\in\; & (\mathbf{a}_{ii}^{\mathfrak{c}})^{-1}\left(\operatorname{opp} \sum_{j \neq i} \mathbf{a}_{ij}^{\mathfrak{c}} x_j + \mathbf{b}_i^{\mathfrak{c}}\right) \\
= \; & (\mathbf{a}_{ii}^{\mathfrak{c}})^{-1}\left(\sum_{j \neq i} \operatorname{opp}\left(\mathbf{a}_{ij}^{\mathfrak{c}} x_j\right) + \mathbf{b}_i^{\mathfrak{c}}\right) \\
= \; & (\mathbf{a}_{ii}^{\mathfrak{c}})^{-1}\left(\sum_{j \neq i} (\operatorname{opp} \mathbf{a}_{ij}^{\mathfrak{c}})\, x_j + \mathbf{b}_i^{\mathfrak{c}}\right) \qquad \text{—since all } x_j \text{ are noninterval} \\
\subseteq \; & (\mathbf{a}_{ii}^{\mathfrak{c}})^{-1}\left(\sum_{j \neq i} (\operatorname{opp} \mathbf{a}_{ij}^{\mathfrak{c}})\, \mathbf{x}_j + \mathbf{b}_i^{\mathfrak{c}}\right) \qquad \begin{array}{l}\text{since } x_j \in \mathbf{x}_j \text{ and interval} \\ \text{— arithmetic operations in } \mathbb{KR} \\ \text{are inclusion monotonic.}\end{array}
\end{aligned}$$

Therefore, if the interval vector $\tilde{\mathbf{x}}$ is determined by the componentwise equalities

$$\tilde{\mathbf{x}}_i := (\mathbf{a}_{ii}^{\mathfrak{c}})^{-1}\left(\sum_{j \neq i} (\operatorname{opp} \mathbf{a}_{ij}^{\mathfrak{c}})\, \mathbf{x}_j + \mathbf{b}_i^{\mathfrak{c}}\right), \qquad i = 1, \ldots, n, \tag{7.12}$$

it thereby

- must be a proper interval, in spite of the possible presence of improper intervals $\mathbf{a}_{ij}^{\mathfrak{c}}$ and $\mathbf{b}_i^{\mathfrak{c}}$ in the expression (7.12),
- provides us with an outer interval estimate of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ too.

So, the natural idea is to take the intersection

$$\mathbf{x} \cap \tilde{\mathbf{x}} \supseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}),$$

which may prove a more narrow estimate than each of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ on its own.

Finally, to make the best use of the information obtained during the runtime, we can, similar to the classical Gauss-Seidel method, immediately involve the new

Table 4. Generalized interval Gauss-Seidel method for outer estimation of the AE-solution sets of interval linear system.

Input

Characteristic matrix $\mathbf{A}^{\mathrm{c}} \in \mathbb{KR}^{n\times n}$ and right-hand side vector $\mathbf{b}^{\mathrm{c}} \in \mathbb{KR}^{n}$ that correspond to the AE-solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ estimated of an interval system $\mathbf{A}x = \mathbf{b}$.

An interval vector $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_n)^\top \in \mathbb{IR}^n$ that bounds the desired portion of the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$.

A prescribed accuracy $\varepsilon > 0$.

Output

Either the information "the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ does not intersect $\mathbf{x}$" or new outer estimate $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1,\ldots,\tilde{\mathbf{x}}_n)^\top \supseteq \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) \cap \mathbf{x}$.

Algorithm

$d \leftarrow +\infty$;
DO WHILE $(d \geq \varepsilon)$
  DO FOR $i = 1$ TO n

$$\tilde{\mathbf{x}}_i \leftarrow (\mathbf{a}_{ii}^{\mathrm{c}})^{-1}\left(\sum_{j=1}^{i-1}(\mathrm{opp}\ \mathbf{a}_{ij}^{\mathrm{c}})\,\tilde{\mathbf{x}}_j + \sum_{j=i+1}^{n}(\mathrm{opp}\ \mathbf{a}_{ij}^{\mathrm{c}})\,\mathbf{x}_j + \mathbf{b}_i^{\mathrm{c}}\right);$$

    IF ($\tilde{\mathbf{x}}_i$ is an improper interval) THEN
      STOP, signaling "the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ does not intersect $\mathbf{x}$"
    END IF
    $\tilde{\mathbf{x}}_i \leftarrow \mathbf{x}_i \cap \tilde{\mathbf{x}}_i$;
    IF ($\tilde{\mathbf{x}}_i = \emptyset$) THEN
      STOP, signaling "the solution set $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})$ does not intersect $\mathbf{x}$"
    END IF
  END DO
  $d \leftarrow$ distance between $\mathbf{x}$ and $\tilde{\mathbf{x}}$;
  $\mathbf{x} \leftarrow \tilde{\mathbf{x}}$;
END DO

estimate of each component (which is at least as good as the old one) into the computation. The i-th component of the new estimate $\tilde{\mathbf{x}}$ is thus to be computed in accordance with the formula (7.12) relying on the already found components of $\tilde{\mathbf{x}}$ with the numbers $1, 2, \ldots, (i-1)$ and the $(i+1)$-st, …, n-th components of the old estimate $\mathbf{x}$.

The overall computational scheme of the interval Gauss-Seidel iteration for computing the enclosures of the AE-solution sets to interval linear systems is presented in Table 4. If $\Xi_{\alpha\beta}(\mathbf{A},\mathbf{b}) \cap \mathbf{x} \neq \emptyset$, then the result of the execution of the algorithm is a sequence $\{\tilde{\mathbf{x}}\}$ of proper nested intervals that must have a limit in

$\mathbb{IR}^n$ (see [2], [45], [69]). The stopping criteria for the above iteration is, as usual, attaining sufficient closeness (in some interval metric) between the two successive approximations.

To start our interval Gauss-Seidel method we need an initial interval vector $\mathbf{x} \supseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. For the generalized solution sets, we can always take it as an enclosure of the united solution set $\Xi_{uni}(\mathbf{A}, \mathbf{b})$ for the corresponding interval linear system (since Ξ_{uni} is the widest among the solution sets), applying any one of the numerous techniques that have been elaborated for this purpose [2], [32], [69].

As a specific example, we consider again the interval linear system

$$\begin{pmatrix} [\ 2,4] & [-1,1] \\ [-1,1] & [\ 2,4] \end{pmatrix} x = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix} \tag{6.7}$$

from [69], [71] and the outer estimation of its $\binom{\exists\forall}{\exists\exists}\binom{\exists}{\exists}$-solution set. Straightforward applying the generalized interval Gauss-Seidel iteration results in

$$\begin{pmatrix} \left[-\frac{3}{2}, \frac{3}{2}\right] \\ \left[-\frac{3}{4}, \frac{3}{4}\right] \end{pmatrix},$$

the tightest possible enclosure for the $\binom{\exists\forall}{\exists\exists}\binom{\exists}{\exists}$-solution set (see Figure 10), which coincides with what we get by the formal approach in Section 7.2.

W. Barth and W. Nuding [6] and afterward A. Neumaier [69], [70] gave a profound investigation of the interval Gauss-Seidel method for the classical case of enclosing the united solution set to interval linear systems. The theory developed by Barth-Nuding and Neumaier can be partly transferred to the generalized interval Gauss-Seidel method we have just derived. That has been done in [109], [110], although changing accents and interpretation of some results as compared with Neumaier's theory [69], [70].

The key point in the considerations of A. Neumaier is the concepts of *H*-matrix:

DEFINITION 7.4 [69], [70]. For a proper interval matrix $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{IR}^{n\times n}$, its *comparison matrix* is termed the matrix $\langle \mathbf{A} \rangle \in \mathbb{R}^{n\times n}$ such that

$$\text{the } ij\text{-th entry of } \langle \mathbf{A} \rangle := \begin{cases} \langle \mathbf{a}_{ij} \rangle, & \text{if } i = j, \\ -|\mathbf{a}_{ij}|, & \text{if } i \neq j. \end{cases}$$

DEFINITION 7.5 [69], [70]. A proper interval square matrix $\mathbf{A}$ is called an *H-matrix*, if its comparison matrix is an *M*-matrix.

In particular, strictly diagonally dominant interval matrices are *H*-matrices.

THEOREM 7.5. *If* $\mathbf{x}^*$ *is the limit of the generalized Gauss-Seidel method applied to an interval linear system* $\mathbf{A}x = \mathbf{b}$, *then*

$$\langle \mathbf{A} \rangle |\mathbf{x}^*| \leq |\mathbf{b}|. \tag{7.13}$$

If $\mathbf{A}$ *is an interval H-matrix, then*

$$|\mathbf{x}^*| \le \langle \mathbf{A} \rangle^{-1} |\mathbf{b}|. \tag{7.14}$$

Proof. We consider only nonsingular interval matrices $\mathbf{A}$, assuming without loss in generality that $0 \notin \mathbf{a}_{ii}$. The formulas specifying the generalized interval Gauss-Seidel method thus imply

$$\mathbf{x}_i^* \subseteq (\mathbf{a}_{ii}^{\mathfrak{c}})^{-1} \left(\sum_{j \ne i} (\operatorname{opp} \mathbf{a}_{ij}^{\mathfrak{c}})\, \mathbf{x}_j^* + \mathbf{b}_i^{\mathfrak{c}} \right),$$

so that

$$|\mathbf{x}_i^*| \le \langle \mathbf{a}_{ii} \rangle^{-1} \left(\sum_{j \ne i} |\mathbf{a}_{ij}|\, |\mathbf{x}_j^*| + |\mathbf{b}_i| \right)$$

since both sides of the above inclusion are proper intervals. We get therefore

$$\langle \mathbf{a}_{ii} \rangle |\mathbf{x}_i^*| \le \sum_{j \ne i} |\mathbf{a}_{ij}|\, |\mathbf{x}_j^*| + |\mathbf{b}_i|,$$

which is equivalent to

$$(\langle \mathbf{A} \rangle\, |\mathbf{x}^*|)_i \le |\mathbf{b}_i|$$

for all $i = 1, 2, \ldots, n$, that is, coincides with (7.13).

If $\mathbf{A}$ is an interval H-matrix, then $\langle \mathbf{A} \rangle$ is an M-matrix, so $\langle \mathbf{A} \rangle^{-1} \ge 0$. Multiplying both sides of (7.13) by $\langle \mathbf{A} \rangle^{-1}$, we arrive at (7.14). □

It follows from the inequality (7.14) that any sufficiently large initial box is improved (i.e., is decreased in size) by the generalized Gauss-Seidel iteration providing that the matrix $\mathbf{A}$ is an H-matrix. On the contrary, if $\mathbf{A}$ is not an H-matrix, we cannot draw such a conclusion. Under these circumstances, A. Neumaier in [69] even proved the following showy result for the classical version of the interval Gauss-Seidel iteration:

THEOREM 7.6 [69]. *If a proper interval* $n \times n$*-matrix* $\mathbf{A} = (\mathbf{a}_{ij})$ *is not an H-matrix, then there exist arbitrary large proper interval vectors that cannot be improved by Gauss-Seidel iteration as applied for outer estimation of the united solution set of the interval system* $\mathbf{A}x = 0$.

For the generalized interval Gauss-Seidel method under study, the above theorem is not valid any longer in case the characteristic matrix $\mathbf{A}^{\mathfrak{c}}$ contains at least one proper interval in each row, i.e. the original interval linear system has at least one A-uncertain interval parameter in each row of the matrix. The reason is simple: the magnitude of the interval product is not equal to the product of the factors' magnitudes in Kaucher complete interval arithmetic.

As an example, one can take the interval linear system

$$\begin{pmatrix} [\ \ 2,4] & [-2,1] \\ [-1,2] & [\ \ 2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \tag{3.21}$$

and the outer estimation of its $\binom{\forall\forall}{\forall\exists}\binom{\exists}{\exists}$-solution set. The interval matrix of the system (3.21) is not an *H*-matrix insofar as its comparison matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

is singular. Still, the generalized interval Gauss-Seidel iterations converge, from any sufficiently large box, to the interval enclosure $([-1,1],[-1,1])^{\top}$ of the $\binom{\forall\forall}{\forall\exists}\binom{\exists}{\exists}$-solution set (see Figure 4). Moreover, when enclosing the same solution set to the interval linear system with the matrix from (3.21) and zero right-hand side vector, the generalized interval Gauss-Seidel method also produces the correct answer—zero vector—starting from any zero-containing initial box.

To resume, the generalized version of the interval Gauss-Seidel iteration may behave itself much better than its classical prototype, which is especially pronounced when the number of the A-uncertain parameters in the matrix becomes substantial.

One of the most remarkable facts with the interval Gauss-Seidel iteration as applied to the united solution set is the following optimality property: *if the matrix of the interval linear system is an interval M-matrix, the method produces the interval hull of the solution set*. This fact has been first revealed by W. Barth and W. Nuding [6]. We managed to generalize this classical result as the following weaker

THEOREM 7.7 [109]. *If, in an interval linear system* $\mathbf{A}x = \mathbf{b}$, *the matrix* $\mathbf{A} = (\mathbf{a}_{ij})$ *is an interval M-matrix and no component of* $\mathbf{b}$ *contain zero in its interior, then the generalized interval Gauss-Seidel iteration applied to this system converges to the interval hull of the AE-solution set.*

7.4. PRECONDITIONING

The techniques we have developed in two preceding sections for the outer estimation of the AE-solution sets to interval linear systems—formal approach and interval Gauss-Seidel iteration—have substantial restrictions on their applicability scopes. The keypoint of the feasibility of the formal approach is the reduction of the original linear system (1.3)–(1.4) to the form (7.6) so that the condition $\rho(|I \ominus \mathbf{A}^{\mathfrak{c}}|) < 1$ is fulfilled. In its turn, a good work of the interval Gauss-Seidel iteration requires that the interval linear system has an *H*-matrix. These conditions are quite burdensome being obviously not always the case. How can we compute outer estimates of the AE-solution sets to interval linear systems in general?

In the classical problem of outer interval estimation of the united solution set the above difficulty is usually overcome by the so-called *preconditioning* first suggested

by E. Hansen and R. Smith [33] (see also [2], [45], [69]*). Similar to the classical computational linear algebra, preconditioning of an interval system amounts to multiplying both its sides, from the left, by a point matrix (often taken as the inverse to the midpoint matrix of the interval system). So, for some $\Lambda \in \mathbb{R}^{n\times n}$, instead of the original system

$$\mathbf{A}x = \mathbf{b} \tag{1.4}$$

we arrive at the *preconditioned interval system*

$$(\Lambda\mathbf{A})\,x = \Lambda\mathbf{b}, \tag{7.15}$$

its united solution set being almost always wider than that for (1.4). On the other hand, the properties of the interval matrix of the preconditioned system improve (see [69]). Unfortunately, this prescription, which we are going to refer to as *naive preconditioning*, cannot be directly applied to the outer estimation of the generalized solution sets.

When simply multiplying, from the left, the interval matrix and right-hand side vector by a point matrix, the generalized solution sets do not necessarily widen, but can change in quite a sophisticated way. To visually demonstrate that, we consider the interval linear system

$$\begin{pmatrix} [\ \ 2,4] & [-2,1] \\ [-1,2] & [\ \ 2,4] \end{pmatrix} x = \begin{pmatrix} [1,2] \\ [1,2] \end{pmatrix}, \tag{7.16}$$

for which

$$\operatorname{mid}\mathbf{A} = \begin{pmatrix} 3 & -\frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix}, \qquad (\operatorname{mid}\mathbf{A})^{-1} = \begin{pmatrix} \frac{12}{37} & \frac{2}{37} \\ -\frac{2}{37} & \frac{12}{37} \end{pmatrix},$$

while the interval system "naively preconditioned" by the midpoint inverse is

$$\tfrac{2}{37}\begin{pmatrix} [\ \ 11,26] & [-10,10] \\ [-10,10] & [\ \ 11,26] \end{pmatrix} x = \tfrac{2}{37}\begin{pmatrix} [7,14] \\ [4,11] \end{pmatrix}. \tag{7.17}$$

We can see from the lower left picture of Figure 13 that the set of $\binom{\forall}{\exists}\binom{\exists}{\exists}$-solutions of the "naively preconditioned" system (7.17) does not contain, in the first orthant, the vertex $\left(\frac{4}{3}, \frac{5}{3}\right)$ and the adjacent part (e.g., the point $(1,1)^\top$) of the set of $\binom{\forall}{\exists}\binom{\exists}{\exists}$-solutions to the original system (7.16). Moreover, the lower bound of the second coordinate of the points from this solution set, which is equal to zero and reached at the vertex $\left(\frac{1}{2}, 0\right)$ for the original system, increases as the result of the naive preconditioning!

To summarize, the solution sets of the "naively preconditioned" interval linear system do not necessarily contain the solution sets of the original interval system,

* Alefeld and Herzberger even call preconditioning "Hansen method" in their book [2, Chapter 16].

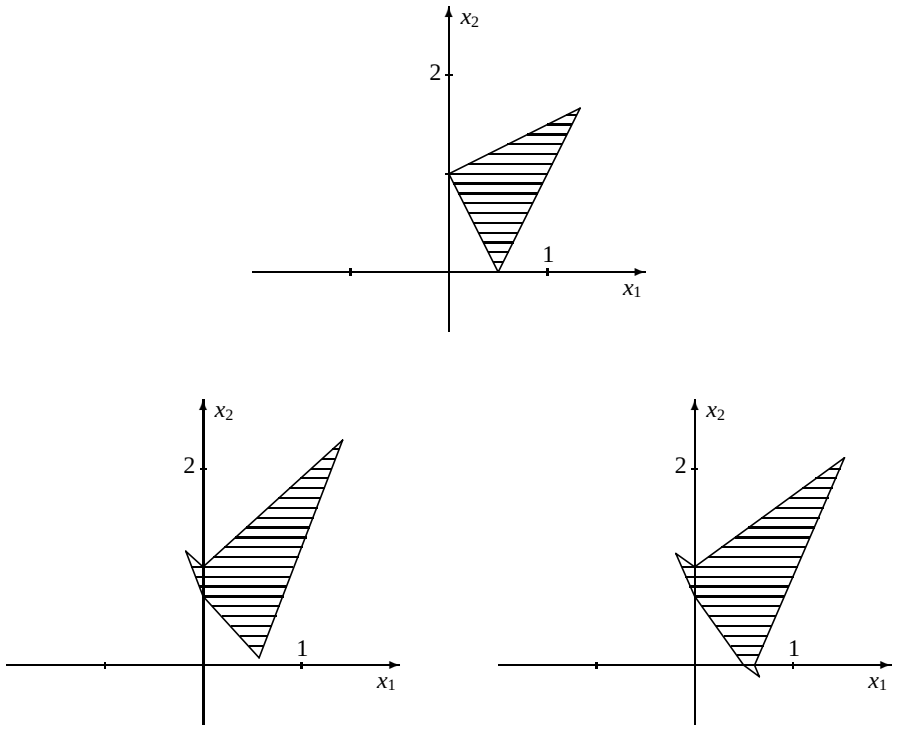

Figure 13. The upper picture shows the set of $\binom{\forall\exists}{\exists}$-solutions for (7.16), while the lower left one is the set of $\binom{\forall\exists}{\exists}$-solutions for (7.17) and the lower right one is the solution set corresponding to the characteristic matrix and right-hand side vector (7.18).

while an outer estimate of the solution set to the "naively preconditioned" interval system may not be an outer estimate of the corresponding solution set for the original interval system. Still, an outcome from the above difficulty does exist and is as follows: *we should precondition the characteristic matrix and characteristic right-hand side vector corresponding to the solution set under consideration* rather than the interval system on its own.

Let us turn again to Theorem 5.1 that gives a very convenient characterization of the AE-solution sets to interval linear systems:

$$x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \quad \Longleftrightarrow \quad \mathbf{A}^{\mathbf{c}} \cdot x \subseteq \mathbf{b}^{\mathbf{c}},$$

$\mathbf{A}^{\mathbf{c}}$ and $\mathbf{b}^{\mathbf{c}}$ being the characteristic matrix and right-hand side that correspond to the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. If Λ is a square point $n \times n$-matrix, then the inclusion $\mathbf{A}^{\mathbf{c}} x \subseteq \mathbf{b}^{\mathbf{c}}$ yields

$$\Lambda(\mathbf{A}^{\mathbf{c}} x) \subseteq \Lambda \mathbf{b}^{\mathbf{c}}.$$

Multiplication of the interval matrices is known to be non-associative in general, but, for point Λ and x, we can avail ourselves of the result of Proposition 5.2:

$$\Lambda(\mathbf{A}^{\mathbf{c}}\, x) = (\Lambda\mathbf{A}^{\mathbf{c}})\, x.$$

Therefore, we arrive at the implication

$$x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \quad \Longrightarrow \quad (\Lambda\mathbf{A}^{\mathbf{c}})\, x \subseteq \Lambda\mathbf{b}^{\mathbf{c}},$$

the sense of which can be interpreted as follows:

THEOREM 7.8. *If $\Lambda \in \mathbb{R}^{n\times n}$ is a square point matrix, then the AE-solution set $\Xi(\mathbf{A}^{\mathbf{c}}, \mathbf{b}^{\mathbf{c}})$ for the interval linear system (1.4), which corresponds to the characteristic matrix $\mathbf{A}^{\mathbf{c}}$ and characteristic right-hand vector $\mathbf{b}^{\mathbf{c}}$, is included in the AE-solution set corresponding to the characteristic matrix $\Lambda\mathbf{A}^{\mathbf{c}}$ and right-hand side vector $\Lambda\mathbf{b}^{\mathbf{c}}$, that is, in $\Xi(\Lambda\mathbf{A}^{\mathbf{c}}, \Lambda\mathbf{b}^{\mathbf{c}})$.*

We shall call multiplying, from the left, both the characteristic matrix and right-hand side vector *generalized preconditioning* of the interval linear system (or just *preconditioning* if that does not confuse). According to Theorem 7.8, it can result only in extending (if any) of the solution set, but the new characteristic interval matrix may satisfy the condition

$$\rho(|I \ominus \mathbf{A}^{\mathbf{c}}|) < 1,$$

which is so crucial for the applicability of our techniques. The initial problem of the outer estimation of an AE-solution set could thus be changed to the other outer estimation problem which corresponds to the preconditioned characteristic matrix and right-hand side and which is computationally tractable.

For example, the characteristic matrix and right-hand side vector of the set of $\binom{\forall}{\exists}\binom{\exists}{\exists}$-solutions to the interval linear system (7.16) are

$$\mathbf{A}^{\mathbf{c}} = \begin{pmatrix} [2, \ \ 4] & [1, -2] \\ [2, -1] & [4, \ \ 2] \end{pmatrix}, \qquad \mathbf{b}^{\mathbf{c}} = \begin{pmatrix} [-2, 2] \\ [-2, 2] \end{pmatrix},$$

so that

$$\begin{aligned} (\operatorname{mid}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{c}} &= \tfrac{2}{37}\begin{pmatrix} [14, \ 23] & [10, -10] \\ [\ 8, -8] & [26, \ \ 11] \end{pmatrix}, \\ (\operatorname{mid}\mathbf{A})^{-1}\mathbf{b}^{\mathbf{c}} &= \tfrac{2}{37}\begin{pmatrix} [-14, 14] \\ [-14, 14] \end{pmatrix}. \end{aligned} \tag{7.18}$$

The AE-solutions that set corresponding to the characteristic matrix and right-hand side vector (7.18) is exhibited at the lower right picture of Figure 13, and it includes all the $\binom{\forall}{\exists}\binom{\exists}{\exists}$-solutions of the original interval linear system (7.16) as one can make sure of from comparison with the upper picture of this figure.

Furthermore,

$$|I \ominus (\operatorname{mid} \mathbf{A})^{-1}\mathbf{A}^{\mathrm{c}}| = \tfrac{1}{37}\begin{pmatrix} 9 & 20 \\ 16 & 15 \end{pmatrix},$$

the eigenvalues of this matrix are equal to $\frac{1}{37}(12 \pm \sqrt{329})$, and the condition (7.5) of Theorem 7.3 is fulfilled indeed allowing one to make use of the formal approach for the outer estimation problem. At the same time, this condition does not hold for the interval matrix of the original system (7.16).

For the reader's convenience, we reformulate the main results of the formal approach of Section 7.2 in the form that explicitly takes into account the preconditioning matrix Λ.

THEOREM 7.9. *Let Λ be a square point matrix. If a point $x \in \mathbb{R}^n$ belongs to the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, then*

$$x \in (I \ominus \Lambda\mathbf{A}^{\mathrm{c}})\, x + \Lambda\mathbf{b}^{\mathrm{c}}.$$

Notice that the above result, as opposed to Theorem 7.1, is only a necessary condition of the membership $x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

THEOREM 7.10. *Let, for an interval linear system $\mathbf{A}x = \mathbf{b}$ and its solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ corresponding to the characteristic matrix $\mathbf{A}^{\mathrm{c}}$ and right-hand side vector $\mathbf{b}^{\mathrm{c}}$, there exist a square point matrix Λ such that*

$$\rho(|I \ominus \Lambda\mathbf{A}^{\mathrm{c}}|) < 1. \tag{7.19}$$

Then a formal solution to the interval linear system

$$x = (I \ominus \Lambda\mathbf{A}^{\mathrm{c}})\, x + \Lambda\mathbf{b}^{\mathrm{c}} \tag{7.20}$$

exists in $\mathbb{KR}^n$ and is unique. If, additionally, the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ is not empty, then the formal solution to the interval system (7.20) is a proper interval vector that contains $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

Proofs of these statements are completely analogous to the proofs of Theorems 7.1–7.3 so that we skip them. □

Preconditioning procedure for the classical case of the united solution set to interval linear systems has been examined by A. Neumaier in [69], [70], and a short summary of Neumaier's theory follows.

THEOREM 7.11 [86]. *If an interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is such that* mid $\mathbf{A}$ *is regular and $|(\operatorname{mid} \mathbf{A})^{-1}| \cdot \operatorname{rad} \mathbf{A}$ has spectral radius < 1, then $\mathbf{A}$ is regular.*

Theorem 7.11 gives thereby only a sufficient condition for the regularity. Nevertheless, the class of interval matrices determined by this condition proves so useful and important that it deserves to be classified as an independent concept:

DEFINITION 7.6 [69]. We shall say that an interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is *strongly regular* (strongly nonsingular) if regular is the matrix mid $\mathbf{A}$ and $\rho(|(\operatorname{mid}\mathbf{A})^{-1}|\operatorname{rad}\mathbf{A}) < 1$.

Every strongly regular matrix is thus just regular (nonsingular), but the reverse is not true. Neumaier matrix (see [69])

$$\begin{pmatrix} 3 & [0,2] & [0,2] \\ [0,2] & 3 & [0,2] \\ [0,2] & [0,2] & 3 \end{pmatrix}$$

may serve as a counterexample. The question on how the regularity and strong regularity relate to each other proved to be not so easy [92], and its complete solution has been recently obtained by A. Lakeyev in [55].

THEOREM 7.12 [69]. *If a matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is strongly regular, then any matrix $\mathbf{B} \subseteq \mathbf{A}$ is also strongly regular.*

THEOREM 7.13 [69]. *Every interval H-matrix (and hence every M-matrix) is strongly regular.*

The class of strongly regular matrices is interesting for us since this is a simply described class of the interval matrices for which the preconditioning enables to attain satisfiability of the conditions (7.5) and (7.19).

DEFINITION 7.7. For a vector $u \in \mathbb{R}^n$, $u > 0$, we shall call *u-scaled maximum-norm* of the matrix $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{KR}^{n\times n}$ the value

$$\|\mathbf{A}\|_u := \max_{1\le i\le n} \left(\frac{1}{u_i} \sum_{j=1}^{n} |\mathbf{a}_{ij}|\, u_j \right).$$

The above definition is a natural extension of the corresponding concept for the point case. As is known from the classical matrix theory (see, e.g., [36], [69]), the spectral radius of a matrix is always no greater than any norm of the matrix. Therefore, for every interval matrix $\mathbf{A} \in \mathbb{KR}^{n\times n}$,

$$\rho(|\mathbf{A}|) \le \| \, |\mathbf{A}| \, \|_u = \|\mathbf{A}\|_u.$$

THEOREM 7.14. [69] *Let the interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ be such that its midpoint matrix* mid $\mathbf{A}$ *is regular. Then the following conditions are equivalent to each other:*

(i) *the matrix* $\mathbf{A}$ *is strongly regular,*

(ii) $\rho\big(|\,(\operatorname{mid}\mathbf{A})^{-1}| \cdot \operatorname{rad}\mathbf{A}\,\big) < 1$,

(iii) $\|I - (\operatorname{mid}\mathbf{A})^{-1}\mathbf{A}\|_u < 1$ *for some* $u > 0$,

(iv) *the product* $(\operatorname{mid}\mathbf{A})^{-1}\mathbf{A}$ *is an H-matrix.*

So, if the interval matrix is strongly regular in the interval linear system under consideration, then we can meet the requirement (7.5) through preconditioning by the midpoint inverse.

THEOREM 7.15. [69] *Let an interval matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *and a point matrix* $\Lambda \in \mathbb{R}^{n\times n}$ *be such that their product* $\Lambda\mathbf{A}$ *is an H-matrix. Then* $\mathbf{A}$ *is strongly regular.*

For the classical problem of outer interval estimation of the united solution set, the most popular choice of the preconditioned matrix is the "midpoint inverse," i.e. $\Lambda = (\text{mid } \mathbf{A})^{-1}$. Such a prescription is very attractive due to some useful properties of the preconditioned matrix: it has the identity matrix as its middle, while optimal bounds of the preconditioned united solution set can be found after only one matrix inversion [31], [87], etc. There are even a number of results showing certain optimality of the preconditioning by the midpoint inverse.

THEOREM 7.16 [70]. *If*

$$\beta := \|I - C\mathbf{A}\|_u < 1$$

in some u-scaled maximum-norm, then β *takes its minimal value for the choice* $C = (\text{mid } \mathbf{A})^{-1}$.

Ris-Neumaier's theory proves to be even excessive for the above developed generalized preconditioning, since the presence of both proper and improper intervals in $\mathbb{KR}$ may lead to decreasing of the width of intervals during arithmetic operations. Anyway, for the strongly regular matrices preconditioning by the midpoint inverse makes the condition (7.5) as well as (7.19) fulfilled, ensuring good work of the interval Gauss-Seidel iteration too.

However, improving the properties of the interval system through preconditioning is not achieved free of charge. The unavoidable price we have to pay for that is the enlargement of the solution set as compared with the original interval system, so that, using preconditioning, we almost always make the outer estimates of the solution set more crude. The amount of such undesirable widening is larger, the greater the preconditioned matrix differ from the diagonal form.

The remarkable fact with the nonsingular diagonal matrix Λ is that there holds the exact equality

$$\Lambda\mathbf{H} = \{\Lambda H \mid H \in \mathbf{H}\},$$

whatever the interval matrix $\mathbf{H}$ of the corresponding size is. For nonsingular diagonal matrices Λ, we can therefore conduct the reasoning of the type

$$H \in \mathbf{H} \text{ is equivalent to } \Lambda H \in \Lambda\mathbf{H}, \tag{7.21}$$

while the preconditioning by such matrices retain the solution set to interval linear system unchanged. In the general case, when Λ is not diagonal nonsingular, we

may realize only the implication from left to right in the logical formula (7.21), and, as a result, the solution set of the interval linear system so preconditioned will not coincide with the original solution set.

On the other hand, having more detailed information on either the interval linear system or the algorithm, it is possible to construct better preconditioners than the midpoint inverse. For example, for the classical interval Gauss-Seidel method even *optimal* (in some natural sense) preconditioners can be constructed, which are recalculated at every algorithm step [45]. As for our formal approach, a promising idea is to consider preconditioners that are intermediate between the pure diagonal and midpoint inverse. Then the solution set would not be distorted much, while still conforming the feasibility condition (7.5). The corresponding rigorous theory is still expecting to be constructed.

7.5. Parameter Partition Method for Interval Linear Systems

In this subsection, we treat the componentwise form (4.5) of the outer estimation problem and concentrate on computing $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})\}$ for a fixed integer index $v \in \{1,2,\ldots,n\}$, since

$$\max\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A},\mathbf{b})\} = -\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A},-\mathbf{b})\}.$$

Let

Encl be a method of outer interval estimation of the AE-solution sets (we shall call it *basic method*),

$\mathrm{Encl}\,(\mathbf{Q},\mathbf{r})$ be an interval outer estimate, produced by the method Encl, of the solution set $\Xi_{\alpha\beta}(\mathbf{Q},\mathbf{r})$ to the system $\mathbf{Q}x = \mathbf{r}$, that is, $\mathrm{Encl}\,(\mathbf{Q},\mathbf{r}) \in \mathbb{IR}^n$ and

$$\mathrm{Encl}\,(\mathbf{Q},\mathbf{r}) \supseteq \Xi_{\alpha\beta}(\mathbf{Q},\mathbf{r}),$$

$\Upsilon(\mathbf{Q},\mathbf{r})$ be the v-th component lower endpoint of the interval enclosure of the solution set $\Xi_{\alpha\beta}(\mathbf{Q},\mathbf{r})$ obtained by the method Encl, that is,

$$\Upsilon(\mathbf{Q},\mathbf{r}) := \underline{\left(\mathrm{Encl}\,(\mathbf{Q},\mathbf{r})\right)}_v. \tag{7.22}$$

We require that the basic method should satisfy the condition:

> The estimate $\Upsilon(\mathbf{Q},\mathbf{r})$ is inclusion monotonic with respect to the matrix $\mathbf{Q}$ and vector $\mathbf{r}$, i.e., for all $\mathbf{Q}'$, $\mathbf{Q}'' \in \mathbb{IR}^{n\times n}$ and $\mathbf{r}'$, $\mathbf{r}'' \in \mathbb{IR}^n$, $\mathbf{Q}' \subseteq \mathbf{Q}''$ and $\mathbf{r}' \subseteq \mathbf{r}''$ implies the inequality
>
> $$\Upsilon(\mathbf{Q}'',\mathbf{r}'') \le \Upsilon(\mathbf{Q}',\mathbf{r}').$$

(7.23)

For the techniques we have developed in the preceding subsections for the outer interval estimation of the AE-solution set to interval linear systems—formal approach and interval Gauss-Seidel iteration—the fulfillment of (7.23) can be easily derived from the inclusion monotonicity of the interval arithmetic operations.

To go further, we need to remind the result of Corollary to Theorem 3.6: both minimal and maximal componentwise values of the points from the solution set are attained at the so-called *extreme* matrices and right-hand side vectors, such that made up of the *endpoints* of **A** and **b**. In other words, for any $v = 1, 2, \ldots, n$,

$$\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\} = (\tilde{A}^{-1}\tilde{b})_v$$

with a point matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ and a point vector $\tilde{b} \in \mathbb{R}^n$ whose elements are the endpoints of the interval entries of the matrix **A** and vector **b** respectively. It is also worth noting that

$$\Upsilon(\tilde{A}, \tilde{b}) \leq (\tilde{A}^{-1}\tilde{b})_v$$

due to the very definition of the estimate Υ. Assuming that an entry $\mathbf{a}_{ij}$ of the matrix **A** has nonzero width, we denote

$$\begin{array}{l} \text{by } \mathbf{A}' \text{ and } \mathbf{A}'' \text{ the matrices obtained from } \mathbf{A} \text{ through} \\ \text{replacing the entry } \mathbf{a}_{ij} \text{ for } \underline{\mathbf{a}}_{ij} \text{ and } \overline{\mathbf{a}}_{ij} \text{ respectively,} \\ \\ \text{by } A' \text{ and } A'' \text{ the matrices obtained from } \tilde{A} \text{ through} \\ \text{replacing the entry } \tilde{a}_{ij} \text{ for } \underline{\mathbf{a}}_{ij} \text{ and } \overline{\mathbf{a}}_{ij} \text{ respectively.} \end{array} \tag{7.24}$$

Inasmuch as

$$A' \subseteq \mathbf{A}' \subseteq \mathbf{A}, \qquad A'' \subseteq \mathbf{A}'' \subseteq \mathbf{A},$$

and $\tilde{b} \subseteq \mathbf{b}$, the condition (7.23) implies the inequalities

$$\Upsilon(\mathbf{A}, \mathbf{b}) \leq \Upsilon(\mathbf{A}', \mathbf{b}) \leq \Upsilon(A', \tilde{b})$$

and

$$\Upsilon(\mathbf{A}, \mathbf{b}) \leq \Upsilon(\mathbf{A}'', \mathbf{b}) \leq \Upsilon(A'', \tilde{b}).$$

Therefore, taking the minima of the corresponding inequality sides, we arrive at

$$\Upsilon(\mathbf{A}, \mathbf{b}) \;\leq\; \min\{\Upsilon(\mathbf{A}', \mathbf{b}),\, \Upsilon(\mathbf{A}'', \mathbf{b})\} \;\leq\; \min\{\Upsilon(A', \tilde{b}),\, \Upsilon(A'', \tilde{b})\}.$$

Additionally,

$$\min\{\Upsilon(A', \tilde{b}),\, \Upsilon(A'', \tilde{b})\} \;\leq\; \Upsilon(\tilde{A}, \tilde{b}) \;\leq\; (\tilde{A}^{-1}\tilde{b})_v \;=\; \min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}.$$

Comparing the two above inequality chains results in the relation

$$\Upsilon(\mathbf{A}, \mathbf{b}) \;\leq\; \min\{\Upsilon(\mathbf{A}', \mathbf{b}),\, \Upsilon(\mathbf{A}'', \mathbf{b})\} \;\leq\; \min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\},$$

and, as a consequence, in the following practical prescription:

Having solved the two interval "systems-descendants" $\mathbf{A}'x = \mathbf{b}$ *and* $\mathbf{A}''x = \mathbf{b}$ *defined by (7.24) we can get better estimate for* $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ *from below as*

$$\min\{\Upsilon(\mathbf{A}', \mathbf{b}),\ \Upsilon(\mathbf{A}'', \mathbf{b})\}.$$

In the right-hand side vector $\mathbf{b}$, breaking an interval element $\mathbf{b}_i$ up into its endpoints $\underline{\mathbf{b}}_i$ and $\overline{\mathbf{b}}_i$ has the similar effect. For uniformity, we will designate by $\mathbf{A}'x = \mathbf{b}'$ and $\mathbf{A}''x = \mathbf{b}''$ the interval "systems-descendants" we get from $\mathbf{A}x = \mathbf{b}$ after having bisected an interval element of either the matrix $\mathbf{A}$ or right-hand side vector $\mathbf{b}$.

To further improve the estimate for $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$, it makes sense to repeat the above described subdivision procedure applying it to the "systems-descendants" $\mathbf{A}'x = \mathbf{b}'$ and $\mathbf{A}''x = \mathbf{b}''$, and then to bisect the descendants of $\mathbf{A}'x = \mathbf{b}'$ and $\mathbf{A}''x = \mathbf{b}''$ again to get even better estimate, and so forth. We arrange the whole process of the successive step-by-step improvement of the estimate for $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ in accordance with the well-known "branch-and-bound" technique, similar to that implemented in the popular interval global optimization methods from [32], [45], [84]:

first, all the interval systems $\mathbf{Q}x = \mathbf{r}$ emerging as the result of the subdivision of the original system (3.14) as well as their estimates $\Upsilon(\mathbf{Q}, \mathbf{r})$ are stored in a *working list* $\mathcal{L}$;

second, at every step of our algorithm, the interval system subject to bisection is that providing the smallest current estimate $\Upsilon(\mathbf{Q}, \mathbf{r})$;

third, the interval element to be bisected in the system $\mathbf{Q}x = \mathbf{r}$ is the one having the maximal width.

The execution of the algorithm thus amounts to maintaining the list $\mathcal{L}$, which consists of records having the form of *triples*

$$\big(\mathbf{Q}, \mathbf{r}, \Upsilon(\mathbf{Q}, \mathbf{r})\big), \tag{7.25}$$

where

$\mathbf{Q}$ is an interval $n \times n$-matrix, $\mathbf{Q} \subseteq \mathbf{A}$,

$\mathbf{r}$ is an interval n-vector, $\mathbf{r} \subseteq \mathbf{b}$.

Besides, the records forming the working list $\mathcal{L}$ will be ordered with respect to the values of the estimate $\Upsilon(\mathbf{Q}, \mathbf{r})$, while the first record of $\mathcal{L}$ as well as the corresponding interval system $\mathbf{Q}x = \mathbf{r}$ and the estimate (the smallest in the list) will be called *leading* ones at the current step of the method. Table 5 summarizes the overall pseudocode of the new algorithm, which we are going to refer to as the simplest *parameter partition method* following the terminology tradition of

Table 5. The simplest PPS-method for interval linear systems.

Input

An interval linear system $\mathbf{A}x = \mathbf{b}$.

An integer index $v \in \{1, 2, \ldots, n\}$.

A method Encl that produces the estimate Υ by the rule (7.22).

Output

An estimate Z for $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ from below.

Algorithm

assign $\mathbf{Q} \leftarrow \mathbf{A}$ and $\mathbf{r} \leftarrow \mathbf{b}$;
compute the estimate $\Upsilon(\mathbf{Q}, \mathbf{r})$, assign $\upsilon \leftarrow \Upsilon(\mathbf{Q}, \mathbf{r})$;
initialize the list $\mathcal{L}$, i.e assign $\mathcal{L} \leftarrow \{(\mathbf{Q}, \mathbf{r}, \upsilon)\}$;

`DO WHILE` (the system $\mathbf{Q}x = \mathbf{r}$ is interval)

in the matrix $\mathbf{Q} = (\mathbf{q}_{ij})$ and vector $\mathbf{r} = (\mathbf{r}_i)$, choose an interval element $\mathbf{h}$ having the maximal width;

generate interval systems $\mathbf{Q}'x = \mathbf{r}'$ and $\mathbf{Q}''x = \mathbf{r}''$ so that

if $\mathbf{h} = \mathbf{q}_{kl}$ for some $k, l \in \{1, 2, \ldots, n\}$, then set

$$\mathbf{q}'_{ij} \leftarrow \mathbf{q}''_{ij} \leftarrow \mathbf{q}_{ij} \quad \text{for } (i,j) \neq (k,l),$$
$$\mathbf{q}'_{kl} \leftarrow \underline{\mathbf{q}}_{kl}, \quad \mathbf{q}''_{kl} \leftarrow \overline{\mathbf{q}}_{kl}, \quad \mathbf{r}' \leftarrow \mathbf{r}'' \leftarrow \mathbf{r};$$

if $\mathbf{h} = \mathbf{r}_k$ for some $k \in \{1, 2, \ldots, n\}$, then set

$$\mathbf{Q}' \leftarrow \mathbf{Q}'' \leftarrow \mathbf{Q}, \quad \mathbf{r}'_k \leftarrow \underline{\mathbf{r}}_k, \quad \mathbf{r}''_k \leftarrow \overline{\mathbf{r}}_k,$$
$$\mathbf{r}'_i \leftarrow \mathbf{r}''_i \leftarrow \mathbf{r}_i \quad \text{for } i \neq k;$$

compute the estimates $\upsilon' \leftarrow \Upsilon(\mathbf{Q}', \mathbf{r}')$ and $\upsilon'' \leftarrow \Upsilon(\mathbf{Q}'', \mathbf{r}'')$;

delete the former leading record $(\mathbf{Q}, \mathbf{r}, \upsilon)$ from the list $\mathcal{L}$;

put the records $(\mathbf{Q}', \mathbf{r}', \upsilon')$ and $(\mathbf{Q}'', \mathbf{r}'', \upsilon'')$ into $\mathcal{L}$ so that the values of the third field of the records in $\mathcal{L}$ increases;

denote the first record of the list $\mathcal{L}$ by $(\mathbf{Q}, \mathbf{r}, \upsilon)$;

`END DO`

$Z \leftarrow \upsilon$;

deterministic global optimization [35]. Another suitable name for the new method is *PPS-method*—after *Partitioning Parameter Set*.* The main idea of this kind of method, first presented by the author in [101], can be partially extended to general nonlinear interval systems of equations, although the result of the bisection of each

* The more so that there exists a dual class of PSS-methods [115], which exploit the idea of *Partitioning Solution Set*.

interval parameter will be *two subintervals* rather than the endpoints as it is in the linear case.

If T is the total number of interval (with nonzero widths) elements in the matrix $\mathbf{A}$ and right-hand side vector $\mathbf{b}$ of the original interval system (3.14) (in general, $T \leq (n+1)n$), then the algorithm of Table 1 stops after at most 2^T steps, producing an estimate for $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ from below. How close the computed result is to the exact value of $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ depends mainly on the way we find the estimate $\Upsilon(\mathbf{Q}, \mathbf{r})$, that is, on the choice of the basic method Encl. In particular, for the computed result to be optimal (exactly equal to $\min\{x_v \mid x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$) it is necessary and sufficient that the following condition holds:

The estimate $\Upsilon(\mathbf{Q}, \mathbf{r})$ is exact for point linear systems

Anyway, if the dimension of the system under solution is sufficiently large and T exceeds mere several tens, then, on modern medium class computers, the simplest parameter partition method will never work till its natural completion, so that it makes good sense to consider it as an *iterative* one.

To conclude, it is worth noting that for the classical problem of outer interval estimation of the united solution set, PPS-method can be further modernized using the information on the monotonicity of the outer estimate and subtle combinatorial properties of the united solution set, which result in extremely efficient numerical algorithms, the best in their class [102].

8. Conclusion

In our work, we have presented a new technique for analyzing and working out the systems under interval uncertainty and ambiguity, its distinctive features being

- a broad use of the logical quantifies and language of the predicate calculus to describe the system's objectives, functioning, operational use and to demarcate between different types of the system inputs and outputs;
- using Kaucher complete interval arithmetic to solve various mathematical problems that arise during the system analysis process.

What are the advantages and benefits of each of the above innovations? And why are they necessary at all? The fact is, applying a formal language to describe a system becomes indispensable when the number of its inputs and outputs grows as well as the requirements on some of these inputs/outputs are not fixed and they can change during the design stage and operational use.

In the problems traditionally considered by classical control theory, the overall number of inputs/outputs is not big, so that working with usual verbal (expressed in words) problem statements encounters no obstacles. The main source of mathematical difficulties, as classical control theory perceives them, is the complex

relationship between the components and parameters of the object under study, e.g., its dynamic character, nonsmoothness of the parameters, instabilities and singularities, etc.

However, if the quantity of inputs/outputs increases, we arrive at a qualitatively new situation when the verbal description becomes hard to write out, understand and operate with. As the result, it is quite easy to mix up which input or output corresponds to this or that uncertainty type, through which input the disturbances act on the system and which ones we can control by ourselves, etc. The latter is especially pronounced for multistage control processes under uncertainty. In these circumstances, the use of a formal language proves inevitable, and in our paper we are simply making this natural step to fill the gap formed in large-scale systems analysis.

As for Kaucher complete interval arithmetic, its promotion also turns out unavoidable as far as we are involved into the study of essentially minimax problems. Additionally, it is an interval arithmetic with good both algebraic and order properties, embracing classical interval arithmetic, so that its wide recognition by the researchers and engineers is only a matter of time.

The proposed formal approach enables us to reduce the problems of inner and outer interval estimation of the generalized solution sets to solving one non-interval equation—dualization equation,—i.e., to a traditional numerical analysis problem. One would naturally like to have this reduction available for the widest possible class of nonlinear mappings F and not only for those with simple occurrences of the control variables as specified in Propositions 6.1–6.3. Enlarging the set of mappings F for which the main problems (4.1) and (4.2) can be solved, either by the formal approach or by its extensions, is an interesting open problem, and some recent advances in its solution can be found in [95].

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