On Nonnegative Interval Linear Systems and Their Solution*

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Abstract

The work considers solution of various problems for interval linear systems with nonnegative matrices. Relying on *shape monotonicity* of the solution sets to such interval systems, we develop a new technique for constructing maximal (with respect to inclusion) inner estimates of the solution sets to nonnegative interval linear equations systems. The main result is extended to general AE-solution sets for interval linear systems.

Keywords: interval linear system, nonnegative matrix, solution set, inner estimation, maximal estimate
AMS subject classifications: 65G40, 65F30

1 Introduction

We consider interval linear systems of equations

$$\left\{ \begin{array}{ccccccccc} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \boldsymbol{a}_{m1}x_1 & + & \boldsymbol{a}_{n2}x_2 & + & \ldots & + & \boldsymbol{a}_{mn}x_n & = & \boldsymbol{b}_m, \end{array} \right. \tag{1}$$

or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b} \tag{2}$$

with an interval matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ and an interval right-hand side vector $\boldsymbol{b} = (\boldsymbol{b}_i)$. The system $\boldsymbol{A}x = \boldsymbol{b}$ is understood as a family of point linear systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. For interval systems of equations various solutions and solution sets can be defined, and the most popular of them is *united solution set*

$$\Xi(\boldsymbol{A}, \boldsymbol{b}) \;=\; \big\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(\, Ax = b\,) \,\big\} \tag{3}$$

formed by all solutions to point linear systems $Ax = b$ with the coefficients $A \in \boldsymbol{A}$ and right-hand sides $b \in \boldsymbol{b}$. Usually, it is called just "solution set", if that does not lead to confusion. In Section 5, we will consider another types of solution sets.

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Exact and complete description of the solution sets is practically impossible due to its enormous complexity, but, on the other hand, it is not necessary in reality. In most cases, it suffices to know an *approximate description*, or *estimate* of the solution set by simpler sets, i.e., having smaller constructive complexity. We are going to touch the problems of its outer estimation —

Find (as tight as possible) interval box that contains
the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ of interval linear system $\boldsymbol{A}x = \boldsymbol{b}$,

which makes sense for bounded solution sets, as well as the problem of its inner estimation —

Find (as wide as possible) interval box that is contained
in the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ of interval linear system $\boldsymbol{A}x = \boldsymbol{b}$.

Practically, *inclusion maximal* inner estimates are most valuable.

In the latter, we do not suppose that the interval matrix $\boldsymbol{A}$ is square, and, in the square case, $\boldsymbol{A}$ need not be regular.

2 Theoretical basis

Our theory starts from the following observation: *if, in the interval linear equations system $\boldsymbol{A}x = \boldsymbol{b}$, all the entries of the matrix $\boldsymbol{A}$ are nonnegative, the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ has monotonic shape.*

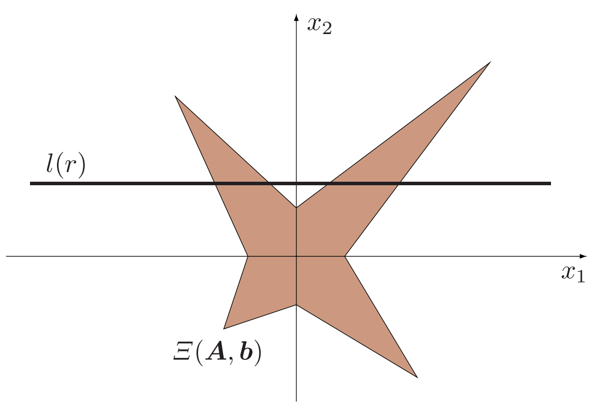

Figure 1: “Axial cut” of the solution set.

Let us specify what is meant by “monotonic shape”. We fix an index $\nu \in \{1, 2, \dots, n\}$

and consider in $\mathbb{R}^n$ a straight line l, with the parametric equation

$$\begin{cases} x_1 &= r_1, \\ & \vdots \\ x_{\nu-1} &= r_{\nu-1}, \\ x_\nu &= t\,, \\ x_{\nu+1} &= r_{\nu+1}, \\ & \vdots \\ x_n &= r_n \qquad (t \in \mathbb{R} \text{ is a parameter}), \end{cases} \tag{4}$$

which is parallel to the ν-th coordinate axis. Every such line is determined by a vector $r \in \mathbb{R}^{n-1}$, $r = (r_1, \dots, r_{\nu-1}, r_{\nu+1}, \dots, r_n)^\top$, and can be denoted as $l(r)$. We define

$$\underline{\Omega}_\nu(r) = \min\{\, x_\nu \mid x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\},$$

$$\overline{\Omega}_\nu(r) = \max\{\, x_\nu \mid x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\}$$

— minimum amd maximum values of the ν-th coordinate of the points from the intersection of $l(r)$ with the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$. $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ will be called *boundary functions* of the solution set along the νth coordinate direction, from below and from above, inasmuch as they really "trace" the boundary of the solution set. Overall, our observation about "monotonicity" of the solution set can be formalized as follows:

Proposition 1. *If the matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ of the interval linear $m \times n$-system $\boldsymbol{A}x = \boldsymbol{b}$ is nonnegative, i.e. all $\boldsymbol{a}_{ij} \geq 0$, then the boundary functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$, $\nu = 1, 2, \dots, n$, are nonincreasing with respect to every variable on their effective domains.*

How can the values of $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ be computed? We can follow the way first outlined in [10].

Let us "substitute" the equation (4) of the line $l(r)$ into the system (1)–(2):

$$\begin{cases} \boldsymbol{a}_{1\nu}\, t + \sum_{j \neq \nu} \boldsymbol{a}_{1j}\, r_j = \boldsymbol{b}_1, \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad \vdots \\ \boldsymbol{a}_{m\nu}\, t + \sum_{j \neq \nu} \boldsymbol{a}_{mj}\, r_j = \boldsymbol{b}_m. \end{cases} \tag{5}$$

The solution set of the i-th equation is

$$\Bigl(\boldsymbol{b}_i - \sum_{j \neq \nu} \boldsymbol{a}_{ij} r_j \Bigr) \,/\, \boldsymbol{a}_{i\nu}, \tag{6}$$

where "/" means division in extended Kahan interval arithmetic [4] that allows zero-containing divisors. We can separately solve each of the one-dimensional equations comprising the system (5), and then intersect m resulting solution sets given by the formula (6). The set $\mathcal{S}$ thus obtained is exactly the set of values of the ν-th coordinate of the points from $\varXi(\boldsymbol{A},\boldsymbol{b})\cap l(r)$. It may be empty, if the system (5) is incompatible, but in any case

$$\underline{\Omega}_\nu(r) \;=\; \min\mathcal{S} \qquad\text{and}\qquad \overline{\Omega}_\nu(r) \;=\; \max\mathcal{S}.$$

If the intervals $\boldsymbol{a}_{i\nu}$, $i=1,2,\ldots,m$, do not contain zero in the interior, in particular, if $\boldsymbol{A}$ is nonnegative, then all the solution sets to one-dimensional equations are *connected* intervals of the form $[p,q]$ or $[-\infty,p]$ or $[q,+\infty]$ or $[-\infty,+\infty]$.

In the points r of the effective domain of the function $\underline{\Omega}_\nu$, there holds

$$\underline{\Omega}_\nu(r) \;=\; \max_{1\le i\le m}\left\{\,\underline{\left(\boldsymbol{b}_i-\sum_{j\ne\nu}\boldsymbol{a}_{ij}r_j\right)\Big/\,\boldsymbol{a}_{i\nu}}\,\right\}.$$

In the points r of the effective domain of the function $\overline{\Omega}_\nu$, there holds

$$\overline{\Omega}_\nu(r) \;=\; \min_{1\le i\le m}\left\{\,\overline{\left(\boldsymbol{b}_i-\sum_{j\ne\nu}\boldsymbol{a}_{ij}r_j\right)\Big/\,\boldsymbol{a}_{i\nu}}\,\right\}.$$

Proof of Proposition 1. It is based on the fact that both lower and upper envelopes of any family of nonincreasing (nondecreasing) functions is nonincreasing (nondecreasing) too.

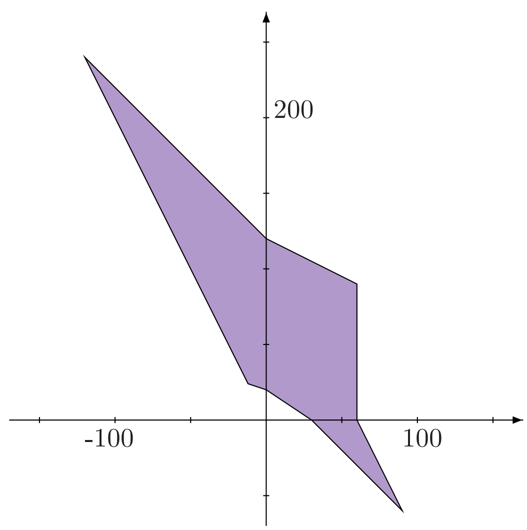

Figure 2: The solution set for Hansen system (8) is not convex, but has monotonic configuration

If $\boldsymbol{A}=(\boldsymbol{a}_{ij})\ge 0$, then, for all i, j and ν, the expressions

$$\frac{\big(\text{endpoint of }\boldsymbol{b}_i\big)-\sum_{j\ne\nu}\big(\text{endpoint of }\boldsymbol{a}_{ij}\big)\;r_j}{\text{endpoint of }\boldsymbol{a}_{i\nu}} \tag{7}$$

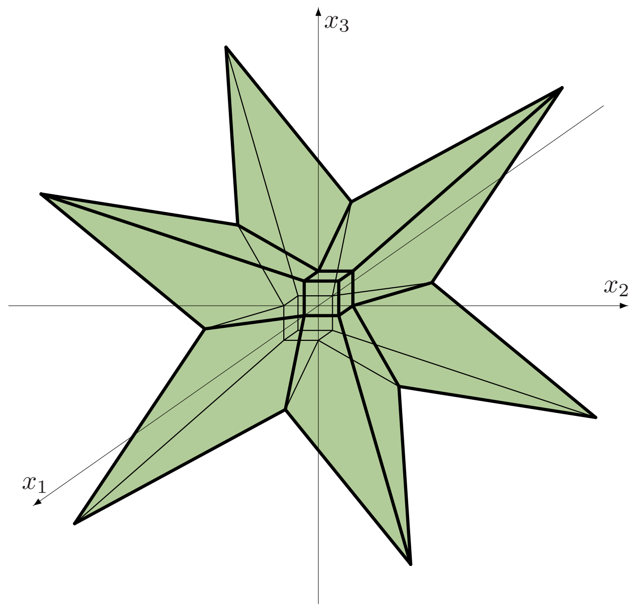

Figure 3: Despite the seemingly chaotic shape, the solution set to Neumaier system (9) is bounded by monotonic surfaces

are monotonically nonincreasing with respect to every argument r_j (providing that the rest arguments are fixed). Therefore, the functions

$$\underline{\omega}_{i\nu}(r) \;=\; \underline{\left(\boldsymbol{b}_i - \sum_{j\neq\nu} \boldsymbol{a}_{ij} r_j\right) / \; \boldsymbol{a}_{i\nu}}, \qquad i = 1, 2, \ldots, m,$$

being the lower envelopes of the functions (7), and the functions

$$\overline{\omega}_{i\nu}(r) \;=\; \overline{\left(\boldsymbol{b}_i - \sum_{j\neq\nu} \boldsymbol{a}_{ij} r_j\right) / \; \boldsymbol{a}_{i\nu}}, \qquad i = 1, 2, \ldots, m,$$

being the upper envelopes of (7), are nonincreasing with respect to r_j.

Since $\underline{\Omega}_\nu(r) = \max_i \underline{\omega}_{i\nu}(r)$ and $\overline{\Omega}_\nu(r) = \min_i \overline{\omega}_{i\nu}(r)$, the proposition follows.

Notice that the boundary functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ may be discontinuous, which is due to zero endpoints of some interval entries in the matrix $\boldsymbol{A}$ of the system. However, if $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is positive, i.e. $\boldsymbol{a}_{ij} > 0$ for every i and j, then the boundary functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$, $\nu = 1, 2, \ldots, n$, are continuous.

The examples illustrating Proposition 1 are well-known Hansen system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}, \tag{8}$$

whose solution set is depicted on Fig. 2, and Neumaier system

$$\begin{pmatrix} 3.5 & [0,2] & [0,2] \\ [0,2] & 3.5 & [0,2] \\ [0,2] & [0,2] & 3.5 \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}, \tag{9}$$

with the solution set as in Fig. 3 (it is shown at the jacket of the book [6], but in another projection).

Discovering nice geometric properties of the solution sets to nonnegative interval linear systems at the end of last century stimulated attempts to construct efficient numerical methods for computing enclosures of these solution sets, such as gradient search along the boundary, etc. However, they all turned out to be unsuccessful, and the issue was closed only after publication of the complexity result [2] stating that both recognition and outer estimation of the solution sets to interval linear systems is NP-hard even if their matrices are positive.

The seeming paradox is explained by the fact that, despite monotonicity of the boundary functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$, their domains may have nonconvex star-shaped configuration (as in Fig. 3), which makes the overall optimization process extremely hard.

Fast outer estimation thus fails, but inner estimation proves more successful.

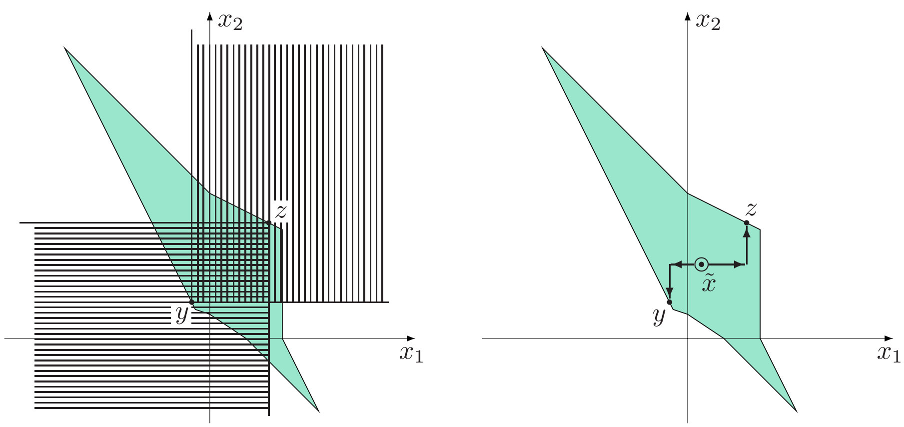

Figure 4: Illustrations of Theorem and of idea of algorithm `INonNeg`.

3 Inner estimation

Theorem. *If, in the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, the matrix $\boldsymbol{A}$ is nonnegative, then for any two points y, $z \in \Xi(\boldsymbol{A}, \boldsymbol{b})$, such that $y \leq z$, the interval box $[y, z]$ is a subset of the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$.*

Proof. It follows from the definition of the boundary functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ that, for any $r \in \mathbb{R}^{n-1}$ and every $\nu \in \{1, 2, \ldots, n\}$, there holds

$$\underline{\Omega}_\nu(r) \;\leq\; \{\, x_\nu \mid x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\} \;\leq\; \overline{\Omega}_\nu(r).$$

Algorithm `INonNeg` for inner estimation
of solution sets to interval linear systems

Input

Interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with the nonnegative matrix.
A point $\tilde{x}$ from the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ under estimation.
Parameters λ, $\mu \in]0, 1]$.

Output

Lower y and upper z bounds of the interval vector $[y, z]$ contained in the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$.

Algorithm

$y \leftarrow \tilde{x}\,;\quad z \leftarrow \tilde{x}\,;$

DO FOR $k = 1$ TO n

  $\boldsymbol{Y} \leftarrow [-\infty, \infty]\,;\quad \boldsymbol{Z} \leftarrow [-\infty, \infty]\,;$

  DO FOR $i = 1$ TO m

$$\boldsymbol{Y} \leftarrow \boldsymbol{Y} \cap \left(\left(\boldsymbol{b}_i - \sum_{j=1, j\neq k}^{n} \boldsymbol{a}_{ij}\, y_j \right) \Big/ \boldsymbol{a}_{ik} \right);$$

$$\boldsymbol{Z} \leftarrow \boldsymbol{Z} \cap \left(\left(\boldsymbol{b}_i - \sum_{j=1, j\neq k}^{n} \boldsymbol{a}_{ij}\, z_j \right) \Big/ \boldsymbol{a}_{ik} \right);$$

  END DO

  IF ($k < n$) THEN

    $y_k \leftarrow \lambda\, \underline{\boldsymbol{Y}} + (1-\lambda)\,\tilde{x}_k\,;\quad z_k \leftarrow (1-\mu)\,\tilde{x}_k + \mu\,\overline{\boldsymbol{Z}}\,;$

  ELSE

    $y_k \leftarrow \underline{\boldsymbol{Y}}\,;\quad z_k \leftarrow \overline{\boldsymbol{Z}}\,;$

  END IF

END DO

If the matrix $\boldsymbol{A}$ is nonnegative, then

$$\left\{\, x_\nu \mid x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\right\} \;=\; \left[\, \underline{\Omega}_\nu(r), \overline{\Omega}_\nu(r) \,\right],$$

since the set $\{\, x_\nu \mid x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\}$ is connected. Therefore, the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is the intersection of the epigraph of $\underline{\Omega}_\nu(r)$ and hypergraph of $\overline{\Omega}_\nu(r)$. The assertion of the theorem stems from the fact that the functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ are nonincreasing.

As the result, we can come up with an algorithm for inner estimation of the solution sets (first published in [12]), whose pseudocode is presented in the table above (the symbol "$\leftarrow$" means assignment operator). It constructs the lower y and upper z bounds of the box $[y, z] \subseteq \Xi(\boldsymbol{A}, \boldsymbol{b})$, starting from a point $\tilde{x} \in \Xi(\boldsymbol{A}, \boldsymbol{b})$. Initially, we assign $y \leftarrow \tilde{x}$ and $z \leftarrow \tilde{x}$, and then the k-th, $k = 1, 2, \ldots, n$, step of the algorithm moves the points y and z apart along the k-th coordinate direction.

Auxiliary scalar parameters λ and μ, $0 < \lambda, \mu \leq 1$, help adjusting the form of the interval estimate $[y, z]$ and its location within the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$. These parameters control the relative values of the shifts of y_k and z_k with respect to $\tilde{x}_k$ during the k-th algorithm step, $k < n$. To ensure that the inner box $[y, z]$ is inclusion maximal, it makes sense to take the points y and z on the boundary of the solution set, and this is why y and z are pushed apart by maximum possible amount at the last n-th step of the algorithm.

4 Choosing initial point

In order to get a solid inner estimate in the algorithm `INonNeg`, it is desirable to have the initial point $\tilde{x}$ lying in the interior $\operatorname{int} \Xi(\boldsymbol{A}, \boldsymbol{b})$ of the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$. In this section, we discuss how to test whether $\tilde{x} \in \operatorname{int} \Xi(\boldsymbol{A}, \boldsymbol{b})$, and how to correct the position of the point $\tilde{x}$.

If the linear system is square and the matrix $\boldsymbol{A}$ is known to be regular, then a point y from $\Xi(\boldsymbol{A}, \boldsymbol{b})$ can be found by solving a system $Ay = b$ with $a \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. But in the most general situation, finding points from the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is not an easy problem, since the recognition of whether $\Xi(\boldsymbol{A}, \boldsymbol{b}) \neq \varnothing$ is NP-complete [2]. Then the following technique based on the so-called "recognizing functional" may prove helpful.

Let $\operatorname{mid} \boldsymbol{a}$ and $\operatorname{rad} \boldsymbol{a}$ be the midpoint and radius of $\boldsymbol{a}$, while $\langle \boldsymbol{a} \rangle$ means the mignitude of $\boldsymbol{a}$, that is, the smallest distance from the points of the interval $\boldsymbol{a}$ to zero:

$$\operatorname{mid} \boldsymbol{a} = \tfrac{1}{2}(\overline{\boldsymbol{a}} + \underline{\boldsymbol{a}}), \qquad \operatorname{rad} \boldsymbol{a} = \tfrac{1}{2}(\overline{\boldsymbol{a}} - \underline{\boldsymbol{a}}), \qquad \langle \boldsymbol{a} \rangle = \begin{cases} \min\{\, |\underline{\boldsymbol{a}}|, |\overline{\boldsymbol{a}}| \,\}, & \text{if } \boldsymbol{a} \not\ni 0, \\ 0, & \text{if } \boldsymbol{a} \ni 0. \end{cases}$$

Then, for an interval $m \times n$-matrix $\boldsymbol{A}$ and an interval m-vector $\boldsymbol{b}$, the expression

$$\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) = \min_{1 \le i \le m} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j \right\rangle \right\}$$

defines a functional $\operatorname{Uni} : \mathbb{R}^n \to \mathbb{R}$, such that the membership of the point x in the solution set of the interval linear systems $\boldsymbol{A}x = \boldsymbol{b}$ is equivalent to nonnegativity of the functional Uni in x:

$$x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \qquad \Longleftrightarrow \qquad \operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) \geq 0.$$

As a consequence, the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ of the interval linear system is "level" set $\{\, x \in \mathbb{R}^n \mid \operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) \geq 0 \,\}$ of the functional Uni.

The functional Uni is not smooth, but it is concave in each orthant of the space $\mathbb{R}^n$, and if the interval matrix $\boldsymbol{A}$ has point columns, then $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b})$ is concave on unions of several orthants. Additionally, the functional $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b})$ attains finite maximum on the whole of the space $\mathbb{R}^n$. If $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) > 0$, then x is a point from the topological interior $\operatorname{int} \Xi(\boldsymbol{A}, \boldsymbol{b})$ of the solution set. Under some mild restrictions upon $\boldsymbol{A}$, $\boldsymbol{b}$ and x, the reverse is also true. Namely, if $\boldsymbol{A}$ does not have zero rows and $\boldsymbol{b}$ does not have degenerate interval (i.e., point) components, then, for any orthant $\mathcal{O}$ from $\mathbb{R}^n$, the membership $x \in \operatorname{int}\big(\Xi(\boldsymbol{A}, \boldsymbol{b}) \cap \mathcal{O}\big)$ implies $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) > 0$.

As the result we naturally arrive to the following procedure for correcting the initial point $\tilde{x}$ of the algorithm `INonNeg`. If $\tilde{x}$ does not satisfy us due to some reasons, then we can use optimization procedures (e.g., the simplest search methods from [1] or subgradient ascent from [13], and so on) to get a better value of the recognizing functional Uni. If the new value $\tilde{x}$ is such that $\operatorname{Uni}(\tilde{x}) > 0$, then we can be sure that $\tilde{x} \in \operatorname{int} \Xi(\boldsymbol{A}, \boldsymbol{b})$.

5 Generalized solution sets

Generalized solution sets originate from the observation that interval uncertainty has *dual character*. Usually, we use an interval $\boldsymbol{v}$ only in connection with a property $P(v)$ that may be fulfilled or not fulfilled for their point members $v \in \boldsymbol{v}$, and

- either the property $P(v)$ holds *for all* $v \in \boldsymbol{v}$,
- or the property $P(v)$ holds *for some* $v \in \boldsymbol{v}$, not necessarily all, maybe, even for only one.

The above distinction between the interpretations of the interval uncertainty is well rendered by logical quantifiers —

- in the first case, we write "$(\forall v \in \boldsymbol{v})\ P(v)$" speaking of *interval uncertainty of A-type*,
- in the second case, we write "$(\exists v \in \boldsymbol{v})\ P(v)$" speaking of *interval uncertainty of E-type*.

As the result, when formulating this or that interval problem statement, we should clearly point out which type of the interval uncertainty corresponds to every interval parameter.

In particular, for an interval system of equations $F(\boldsymbol{a}, x) = \boldsymbol{b}$, the most general definition of the solution set looks like

$$\left\{\, x \in \mathbb{R}^n \mid (\, Q_1 v_{\pi_1} \in \boldsymbol{v}_{\pi_1}) \cdots (\, Q_{l+m} v_{\pi_{l+m}} \in \boldsymbol{v}_{\pi_{l+m}})(\, F(a, x) = b\,)\,\right\}, \tag{10}$$

where $Q_1, Q_2, \ldots, Q_{l+m}$ are logical quantifiers "$\forall$" or "$\exists$",

$(\, v_1, v_2, \ldots, v_{l+m}) := (\, a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_m) \in \mathbb{R}^{l+m}$ is aggregated vector of the parameters of the system,

$(\, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{l+m}) := (\, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_l, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \in \mathbb{IR}^{l+m}$ — is aggregated vector of intervals of their values,

$(\, \pi_1, \pi_2, \ldots, \pi_{l+m})$ is a permutation of the integers $1, 2, \ldots, l+m$.

The solution sets (10) are called *generalized solution sets* to the interval system of equations $F(\boldsymbol{a}, x) = \boldsymbol{b}$ [11].

The definition of generalized solution sets is extremely general. It makes sense to somehow restrict ourselves, and in what follows we are going to consider only the solution sets for which all the occurrences of the universal quantifier "$\forall$" precede those of the existential quantifier "$\exists$" in the logical formula that stands after the vertical line in (10) and "selects" points of the solution set (it is usually called *selecting predicate*). Generalized solution sets to interval equations systems for which the selecting predicate has such special AE-form will be referred to as *AE-solution sets* or *sets of AE-solutions* [11]. For interval linear systems of equations, the above formulated constructions are concretized as follows.

Let, for an interval linear $m \times n$-system $\boldsymbol{A}x = \boldsymbol{b}$, a quantifier $m \times n$-matrix α and a quantifier m-vector β be given as well as associated decompositions of the index sets of the matrix and vector of the same sizes to nonintersecting subsets $\hat{\Gamma} = \{\, \hat{\gamma}_1, \ldots, \hat{\gamma}_p\}$ and $\check{\Gamma} = \{\, \check{\gamma}_1, \ldots, \check{\gamma}_q\}$, $p + q = mn$, $\hat{\Delta} = \{\, \hat{\delta}_1, \ldots, \hat{\delta}_r\}$ and $\check{\Delta} = \{\, \check{\delta}_1, \ldots, \check{\delta}_s\}$, $r + s = m$.

The set

$$\begin{aligned} &\Xi_{\alpha\beta}(\boldsymbol{A},\boldsymbol{b}) \;:= \\ &\Big\{\; x\in\mathbb{R}^n \mid \\ &\quad (\forall a_{\hat\gamma_1}\in\boldsymbol{a}_{\hat\gamma_1})\;\cdots\;(\forall a_{\hat\gamma_p}\in\boldsymbol{a}_{\hat\gamma_p})\;\;(\forall b_{\hat\delta_1}\in\boldsymbol{b}_{\hat\delta_1})\;\cdots\;(\forall b_{\hat\delta_r}\in\boldsymbol{b}_{\hat\delta_r}) \\ &\quad (\exists a_{\check\gamma_1}\in\boldsymbol{a}_{\check\gamma_1})\;\cdots\;(\exists a_{\check\gamma_q}\in\boldsymbol{a}_{\check\gamma_q})\;\;(\exists b_{\check\delta_1}\in\boldsymbol{b}_{\check\delta_1})\;\cdots\;(\exists b_{\check\delta_s}\in\boldsymbol{b}_{\check\delta_s}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\,Ax=b\,)\;\Big\} \end{aligned}$$

is referred to as *AE-solution set of the type* $\alpha\beta$ (or *set of AE-solutions of the type* $\alpha\beta$) to the interval linear system $\boldsymbol{A}x=\boldsymbol{b}$.

Equivalently, AE-solutions sets can be defined as

$$\Xi_{\alpha\beta}(\boldsymbol{A},\boldsymbol{b}) := \big\{\, x\in\mathbb{R}^n \mid (\forall\hat A\in\boldsymbol{A}^{\forall})(\forall\hat b\in\boldsymbol{b}^{\forall}) \\ (\exists\check A\in\boldsymbol{A}^{\exists})(\exists\check b\in\boldsymbol{b}^{\exists})(\,(\,\hat A+\check A)\,x=\hat b+\check b\,)\,\big\},$$

where $\boldsymbol{A}=\boldsymbol{A}^{\forall}+\boldsymbol{A}^{\exists}$ and $\boldsymbol{b}=\boldsymbol{b}^{\forall}+\boldsymbol{b}^{\exists}$ are corresponding disjunct decompositions of the matrix and of the right-hand side vector of the system, i.e. such that, for the matrices $\boldsymbol{A}^{\forall}=(\boldsymbol{a}_{ij}^{\forall})$ and $\boldsymbol{A}^{\exists}=(\boldsymbol{a}_{ij}^{\exists})$, there holds $\boldsymbol{a}_{ij}^{\forall}\boldsymbol{a}_{ij}^{\exists}=0$ and, for the vectors $\boldsymbol{b}^{\forall}=(\boldsymbol{b}_i^{\forall})$ and $\boldsymbol{b}^{\exists}=(\boldsymbol{b}_i^{\exists})$, there holds $\boldsymbol{b}_i^{\forall}\boldsymbol{b}_i^{\exists}=0$.

Notice that the united solution set (3) to the interval linear systems (1)–(2) is an AE-solution set too.

It turns out that, for the interval linear equations system $\boldsymbol{A}x=\boldsymbol{b}$ whose entries of the matrix $\boldsymbol{A}$ are nonnegative, AE-solution sets $\Xi_{\alpha\beta}(\boldsymbol{A},\boldsymbol{b})$ have *monotonic shape.* More precisely, if we redefine the boundary functions

$$\begin{aligned} \underline{\Omega}_\nu(r) &= \min\big\{\, x_\nu \mid x\in\Xi_{\alpha\beta}(\boldsymbol{A},\boldsymbol{b})\cap l(r)\,\big\}, \\ \overline{\Omega}_\nu(r) &= \max\big\{\, x_\nu \mid x\in\Xi_{\alpha\beta}(\boldsymbol{A},\boldsymbol{b})\cap l(r)\,\big\}, \end{aligned}$$

then there holds

Proposition 2 *If the matrix $\boldsymbol{A}$ of the interval linear system $\boldsymbol{A}x=\boldsymbol{b}$ is nonnegative, then the boundary functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$, $\nu=1,2,\dots,n$, are nonincreasing with respect to every variable on their effective domains.*

The proof is very similar to that for Proposition 1 and can be found in [8]. Fig. 5 illustrates our result.

Unfortunately, we cannot avail ourselves of this monotonicity for enclosing AE-solution sets, since A.V. Lakeyev in [3] proved that, in case of "sufficiently many" existential quantifiers "∃", both recognition and outer estimation of AE-solution sets to interval linear systems is NP-hard even if their matrices are strictly positive.

As for inner estimation, Theorem of Section 3 remains valid for AE-solution sets, and the algorithm `INonNeg`, properly adapted, is readily applicable for computing inner boxes of AE-solution sets to nonnegative interval linear systems. Unfortunately, choosing an initial point $\tilde x$ is much harder problem now.

Still, there exists a special case of easily recognizable AE-solution set. It is *tolerable solution set*

$$\Xi_{tol}(\boldsymbol{A},\boldsymbol{b}) = \big\{\, x\in\mathbb{R}^n \mid (\forall A\in\boldsymbol{A})(\exists b\in\boldsymbol{b})(Ax=b)\,\big\},$$

formed by all such points x that the product Ax falls into the right-hand side box $\boldsymbol{b}$ for every $A\in\boldsymbol{A}$. A detailed survey and extensive references on the subject can be

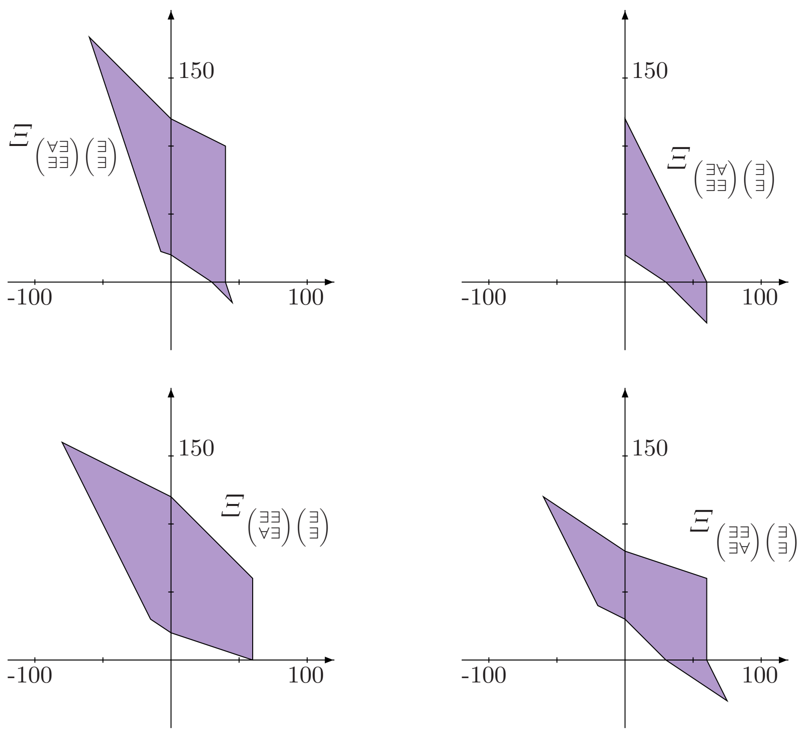

Figure 5: Some AE-solution sets for Hansen system (8)

found in [9]. In particular, in automatic control theory, tolerable solution set and its inner estimation are used in modal regulator synthesis under interval uncertainty [14] and in investigation of asymptotic stability of intervally determined systems [15].

The following characterization of the tolerable solution set $\varXi_{tol}(\boldsymbol{A}, \boldsymbol{b})$ is known:

Rohn theorem [7]. *A point $x \in \mathbb{R}^n$ belongs to the tolerable solution set of interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ if and only if $x = x' - x''$ for vectors x', $x'' \in \mathbb{R}^n$ that satisfy the linear inequalities system*

$$\left\{ \begin{array}{rcr} \overline{\boldsymbol{A}}x' - \underline{\boldsymbol{A}}x'' & \leq & \overline{\boldsymbol{b}}, \\ -\underline{\boldsymbol{A}}x' + \overline{\boldsymbol{A}}x'' & \leq & -\underline{\boldsymbol{b}}, \\ x',\, x'' & \geq & 0. \end{array} \right. \tag{11}$$

Therefore, tolerable solution sets for interval linear systems are *convex polyhedral sets*, and finding a starting point for the algorithm `INonNeg` amounts to searching for an interior point from the solution set to the linear inequalities systems (11). Efficient numerical methods for the solution of this problem are very developed nowadays, and one can look at the related references e.g. in [5]. Next, we can compute inclusion maximal inner interval estimates of the tolerable solution set to an interval linear system with nonnegative matrix by an analog of algorithm `INonNeg` for polynomial time.

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